# The Hydra Game: Winning Without a Torch

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**Abstract**

The mythical Hydra grows two heads for every one cut. In the Kirby–Paris game, it grows entire copies of the branch. Surprisingly, the battle always ends — but the proof requires ordinals beyond infinity, and the statement itself is unprovable in Peano arithmetic. This essay explains the game, the ordinal assignment, and why it gives a natural example of Gödel incompleteness.

## Introduction

The Lernaean Hydra was a creature of terror. It crawled out of the swamp to kill men and cattle, and no sword could master it. It had nine heads, each breathing deadly poison. Heracles, in his second labour, faced it head-on. He chopped off heads one by one, but for each head cut, two new ones grew immediately. The more he fought, the stronger the monster became.

Then Heracles understood that brute force would not work. He called to his nephew Iolaus, who lit a torch. Now every sword stroke was followed by fire. Cauterising the necks stopped the new heads from sprouting. Thus cunning and flame defeated what could not be conquered by strength alone. The Hydra fell, and its immortal head was buried under a huge rock.

Now let me ask you a question, though it may seem unrelated to the story above:

**What do you see in this picture?**

At first glance you might think: what does Greek mythology have to do with... lines? Try to guess.

If you said "tree crown" — wrong. If you said "scheme of a recursive function" — you are probably a snob. If you said "my lecture notes in analysis" — close, but no. But if you said **"several copies of the same branch growing from one node"** — you are almost there.

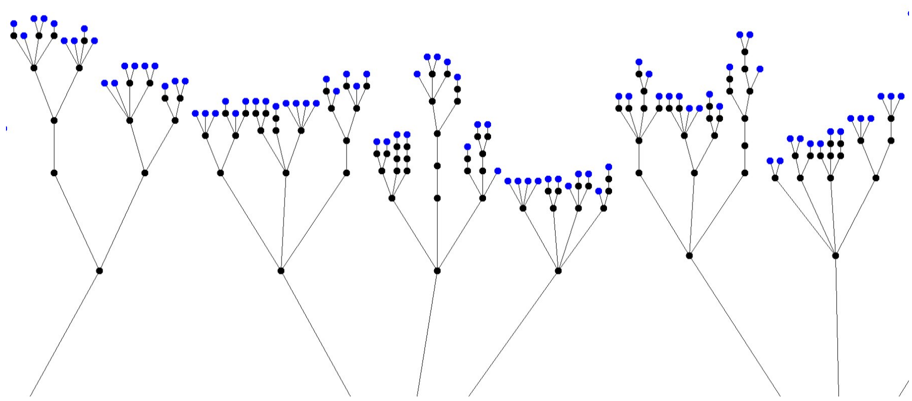

Figure 1: Enter Caption

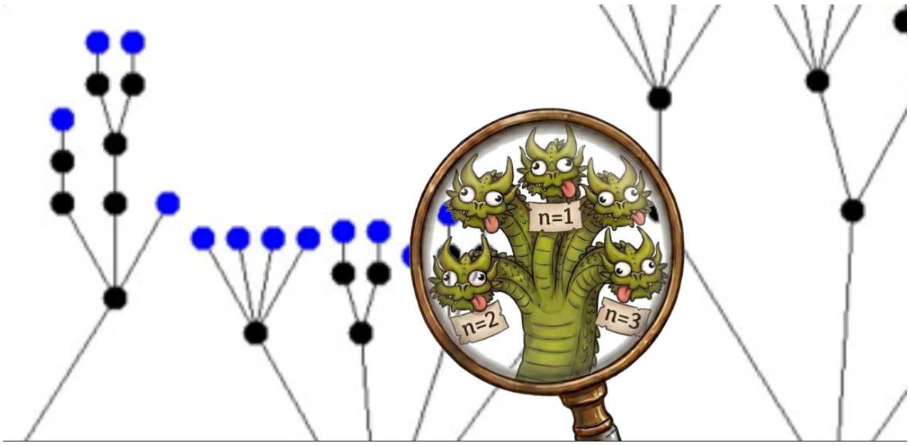


Figure 2: Enter Caption

Drumroll. **This is the mathematical Hydra.** Unlike its mythical cousin, it does not grow two heads for one — it grows whole copies of branches. The longer you play, the larger it becomes. Defeat seems inevitable. Yet the Kirby–Paris theorem (1982) says: victory is guaranteed. Simply because every Hydra can be assigned an ordinal that strictly decreases at each step. And ordinals cannot decrease forever.

The story of this game did not begin in ancient Greece but in Oxford in the late 1970s. Mathematicians Kirby and Paris rewrote the myth. In their version, Heracles forgot his torch at home, and his nephew stayed home making soup. He has only a sword. And the new rule: when you cut off one head, the Hydra grows not two, but **an entire copy of the whole branch on which that head grew**. The battle must end. After a hundred moves. Or a million. Or a number that would not fit in the universe. But it will end. This is the only myth where hope rests not on strength or cunning, but on the properties of transfinite numbers.

In 1982 a paper appeared in the *Bulletin of the London Mathematical Society*. There were no cute Hydras with eyes in that paper — I added those for you. Because a dry article from the London Mathematical Society is all very well, but where is the fun? The fun is this: you take a piece of paper, draw a tree with three leaves, start cutting according to the rules, and after twenty moves your leaf looks like the Tokyo metro map. You are losing. But the theorem says that if you had not given up, you would have won. Eventually.

## 1 Rules of the game

Before we start cutting, let us agree on how we draw it. The root is the "body", leaves are the heads.

```
    r (root)
    |
    a
   / \
  b   c   <- heads (leaves)
```

A simple Hydra: one internal node $a$ and two leaves $b$ and $c$.

## 2 How a Move Works (Diagram 2)

Let us look at our simple Hydra. Cut leaf $b$.

**Before the move:**

```
    r
    |
    a
   / \
  b   c
  ^
cut here
```

**After the move ($N = 1$):** delete $b$. Look at its parent $a$, and then at the grandparent $r$. Take the subtree rooted at $a$ (without $b$) — it now contains only leaf $c$. Copy that subtree once and attach it to $r$.

```
     r
    / \
   a   a'   (a' is a copy)
   |   |
   c   c'
```

We now have two heads ($c$ and $c'$) instead of one. The number of heads increased! Yet, as we shall see, the ordinal decreased.

How is this proved? And why cannot ordinary arithmetic prove it?

To prove that the battle is finite we need **ordinals** — numbers that can count beyond infinity. They look like this:

$$0, 1, 2, 3, \ldots, \omega, \omega + 1, \omega + 2, \ldots, \omega \cdot 2, \omega \cdot 3, \ldots, \omega^2, \omega^3, \ldots, \omega^\omega, \ldots, \varepsilon_0, \ldots$$

The key property: when you cut a head, the ordinal decreases, even though the number of heads may grow. And there are no infinite descending chains of ordinals. Therefore the game always ends.

## 3 How to Turn a Tree into an Ordinal (Diagram 3)

Now we learn to assign an ordinal to every Hydra tree. Define a function $\varphi$(tree).

The rule is simple:

- For the empty tree (root only), $\varphi = 0$.
- The root has children. For each child subtree $T_i$ compute its ordinal. Then **sum** $\omega$ raised to those ordinals:
$$\varphi = \omega^{\varphi(T_1)} + \omega^{\varphi(T_2)} + \cdots + \omega^{\varphi(T_k)}.$$

**Examples :**

**Case 1.** A single leaf on the root:

```
r
|
L
```

A leaf has no children $\Rightarrow \varphi(\text{leaf}) = 0$. Then $\varphi(\text{tree}) = \omega^0 = \mathbf{1}$.

**Case 2.** Two leaves on the root:

```
  r
 / \
L1 L2
```

$\varphi = \omega^0 + \omega^0 = 1 + 1 = \mathbf{2}$.

**Case 3.** A branch of depth 2:

```
r
|
A
|
L
```

First $\varphi(A) = 1$ (as in Case 1). Then $\varphi(\text{tree}) = \omega^1 = \omega$.

This is why depth 2 gives an infinite ordinal: even one such "long" leaf is already infinitely more complex than any finite number of heads.

**Case 4.** Node $A$ with two leaves under the root:

```
  r
  |
  A
 / \
L1 L2
```

$\varphi(A) = \omega^0 + \omega^0 = 2$. Then $\varphi(\text{tree}) = \omega^2 = \boldsymbol{\omega^2}$.

Do you see the pattern? The deeper and bushier the tree, the larger the ordinal.

# 4 Lemma: Each Move Decreases the Ordinal (Diagram 4)

Now the most important part. When you cut a leaf and add copies of the subtree, the ordinal **strictly decreases**. Even though the number of heads may increase.

I will not give a full proof (it requires transfinite induction), but I will show an example of how it works.

Take the tree from Case 3: root $\to$ node $A \to$ leaf. Its ordinal is $\omega$.

**Before the move:**

```
r
|
A
|
L
```

Cut the leaf (first move, $N = 1$). Parent is $A$, grandparent is the root. Add 1 copy of the subtree of $A$ without the leaf. That subtree is just an empty vertex $A$ (ordinal 0). The copy is another empty vertex.

**After the move:**

```
     r
    / \
   A   A'   (A and A' are empty, no leaves)
```

New tree: two empty children of the root. Ordinal $= \omega^0 + \omega^0 = 1 + 1 = \mathbf{2}$.

Compare: we had $\omega$, now 2. $\omega$ is larger than any natural number, so $2 < \omega$. Decrease!
**Another example:** a tree with $\varphi = \omega^2$ (node $A$ with two leaves under the root). After cutting one leaf we obtain a tree with $\varphi = \omega \cdot 2$ (two branches each with one leaf). $\omega \cdot 2 = \omega + \omega$, while $\omega^2 = \omega \cdot \omega$. Since $\omega \cdot \omega > \omega + \omega$ (for $\omega > 2$), the ordinal decreased.

In general the proof uses the fact that replacing a subtree by several copies turns $\omega^\alpha$ into $\omega^{\alpha-1} \cdot N$ (for successor ordinals) or into a sum of smaller terms. The crucial point: the result is always **smaller** than the original.

## 5 Why Victory is Inevitable (and the Connection to Infinite Descent)

So, we have assigned an ordinal to every Hydra tree. We have proved that at each move the ordinal strictly decreases. And there are no infinite descending sequences of ordinals (the well-ordering property). Hence the sequence of moves cannot be infinite. Therefore the game **must** end after finitely many steps. The theorem is proved.

This reasoning is a classic example of **transfinite induction**, which allows us to prove statements for all ordinals, not only for natural numbers.

## 6 How Many Moves? (Spoiler: A Lot)

One can ask a practical question: all right, the game ends, but after how many moves?

The answer depends on strategy. If you cut heads arbitrarily, it may take 100 moves. If you deliberately choose the most harmful heads (those that cause maximal growth), the number of moves becomes monstrous.

For a small Hydra — one leaf at depth 3 (root → A → B → leaf) — the worst-case number of moves is **greater than Graham's number**. Graham's number is the famous enormous number that does not fit in the universe even if you write each digit in a Planck volume. For a slightly more complex Hydra (ordinal $\varepsilon_0$) the number of moves is no longer expressible by any recursive function provably total in Peano arithmetic.

Mathematicians describe such insane growth rates using the **Hardy hierarchy**:

$$\begin{aligned} H_0(n) &= n, \\ H_{\alpha+1}(n) &= H_\alpha(n+1), \\ H_\lambda(n) &= H_{\lambda[n]}(n) \quad \text{for limit } \lambda. \end{aligned}$$

For a Hydra with ordinal $\alpha$ the maximal number of moves is $H_\alpha(1)$. For $\alpha = \omega^2$ this is already a huge function, for $\alpha = \omega^\omega$ it is even larger, and for $\alpha = \varepsilon_0$ it grows faster than anything provably total in PA.

## 7 Unprovability in Peano Arithmetic (Why This Matters)

Now — the most interesting part. The statement "Every Hydra dies after a finite number of moves" is **true**. We have just proved it using ordinals and transfinite induction.

However, **proving this statement while staying inside ordinary Peano arithmetic (PA) is impossible**. Why?

Imagine a person who only knows addition. You try to explain multiplication: "$2 \times 3$ is $2 + 2 + 2$". They understand. But when you say "$a \times b$ for any $a, b$" they cannot grasp the general rule because they lack the tool (multiplication). PA is like such a person. It can prove statements about natural numbers using ordinary induction. But transfinite induction up to $\varepsilon_0$ is a tool that is **stronger** than PA. The fact that $\varepsilon_0$ is well-founded cannot be proved inside PA.

In 1936 Gerhard Gentzen showed that transfinite induction up to $\varepsilon_0$ is equivalent to the consistency of PA. By Gödel's second incompleteness theorem, PA cannot prove its own consistency (if it is indeed consistent). Consequently, PA cannot prove that for every Hydra the game is finite.

This was the **first natural example** of Gödel incompleteness — without artificial self-referential sentences like "this statement is unprovable". Just a game that any schoolchild can play, and a statement that is true but unprovable in PA.

## 8 Concrete Example for a Small Hydra

To be concrete, let us walk through one small example from beginning to end.

**Hydra $T_0$:** root $r$, one child $a$, and $a$ has two leaves $b$ and $c$ (a two-pronged branch at depth 2).

1. Compute $\varphi(T_0)$:

- $a$ has two leaves $\Rightarrow \varphi(\text{subtree } a) = \omega^0 + \omega^0 = 2$.
- Then $\varphi(T_0) = \omega^2 = \omega^2$.

2. **Move 1 ($N = 1$):** Cut leaf $b$. Parent $a$, grandparent $r$. Subtree of $a$ without $b$ leaves one leaf $c$ ($\varphi = 1$). Add 1 copy of that subtree. After the move the root has: original $a$ (with leaf $c$) and new $a'$ (with leaf $c'$).

$$\varphi(T_1) = \omega^1 + \omega^1 = \omega + \omega = \omega \cdot 2.$$

Compare: $\omega^2 > \omega \cdot 2$ — decreased.

3. **Move 2 ($N = 2$):** Cut leaf $c$ (under $a$). Now subtree $a$ becomes empty ($\varphi = 0$). Add 2 copies of the empty subtree (i.e., two empty vertices). At the root: $a$ (empty), $a'$ (with leaf $c'$), and two new empty vertices $a'', a'''$.

$$\varphi(T_2) = \omega^0 + \omega^1 + \omega^0 + \omega^0 = 1 + \omega + 1 + 1 = \omega + 3$$

(since $1 + \omega = \omega$). Previously $\omega \cdot 2 = \omega + \omega$, now $\omega + 3$ — again a decrease.

4. Subsequent moves gradually eat the remaining leaves and empty vertices. After a finite number of steps only the root remains.

Even for such a simple Hydra the process is not instantaneous, but it is guaranteed to terminate.

## 9 Conclusion

So, what have we learned?

- The mathematical Hydra can be defeated even without a torch.
- The secret weapon is ordinals, which decrease at every move.
- The number of moves can be unimaginably huge — larger than Graham's number.
- The statement of victory is **true** but **unprovable** in Peano arithmetic. This makes the Hydra game one of the most beautiful examples of Gödel incompleteness.

Returning to the picture at the beginning: the Hydra is smiling. It thinks it is growing and nothing can stop it. But it does not know that the ordinals have already started to decrease. And somewhere out there, in the transfinite distance, zero is already waiting for it.

Welcome to the game. The sword is in your hands. No torch needed.

*Postscript for those who made it to the end: the Kirby–Paris theorem is a reminder that mathematics is full of statements we can* ***see*** *with our minds but cannot* ***prove*** *within a given formal system. This is not a bug, it is a feature. As Gödel said, "incompleteness is not a limitation of reason, but a limitation of formalism."*

**References**

## References


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