Imagining Imaginary Numbers

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Introduction

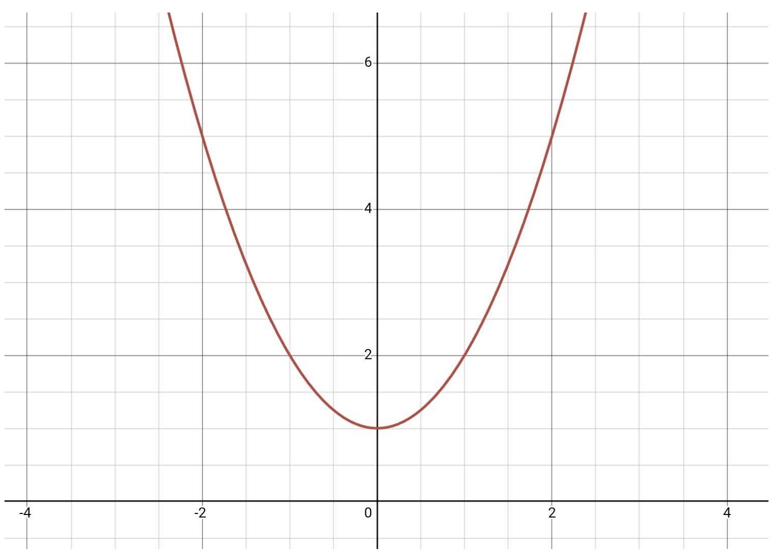
Imagine you are back in 8th grade and your teacher tells you to solve at first glance what looks like an easy question. Find the solutions for the equation:

$$x^2 + 1 = 0$$

Simple right? You just take the 1 to the other side, and square root both sides.

$$x^2 = -1$$
$$x = \sqrt{-1}$$

But wait, what is the square root of negative numbers? Now plugging this into your calculator would have given you a “Math Error”. So does this mean it has no roots? Let's try a different approach and sketch the graph of $y = x^2 + 1$.



Sketching the graph we observe that it lies entirely above the x-axis therefore it has no real solutions when y = 0. But this goes against the fundamental theorem of algebra which states that a polynomial of degree n must have exactly n solutions. So what exactly is going wrong here. Well to understand this we need to take a quick history lesson.

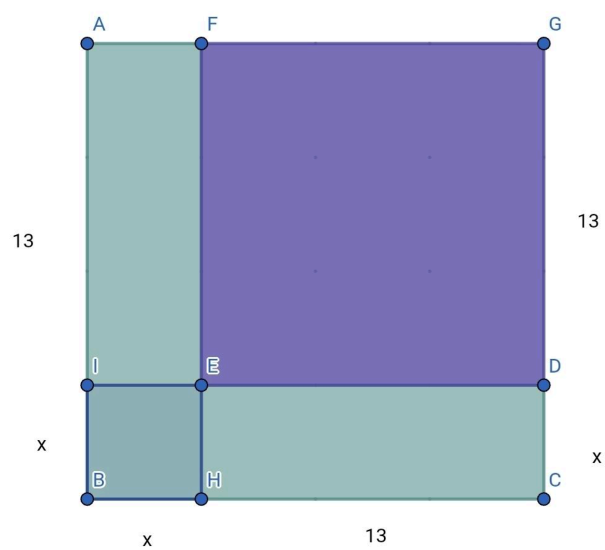
Ancient times

Ancient mathematicians would think of x^2 as a literal square with side lengths x,and x^2 would be the area of that shape. So if they were presented by an equation lets say

$$x^2 + 26x = 27$$

They would think of it as having a square of side lengths x and a rectangle with side lengths x and 26. Now adding the area of both these shapes gives us a total area of 27. So how do they solve for x?

Firstly they would split up the rectangle into 2 smaller ones with lengths x and 13.Now they would rearrange the 3 shapes so that they almost make a square except there is a square missing within it which would then complete the square.



Observing the diagram we can see that the missing square has a length of 13. So we add the area of this square to both sides of the equation.Our new equation is

$$x^2 + 26x + 169 = 27 + 169$$
$$x^2 + 26x + 169 = 196$$
$$(x + 13)^2 = 14^2$$
$$x + 13 = 14$$
$$x = 1$$

So our solution is 1? Well if we refer back to our fundamental theorem of algebra,we know we must end up with 2 solutions as this is a degree 2 polynomial.The second solution which could be found with a little bit of trial and error is -27.But for centuries mathematician ignored this negative solution as they thought this made no sense.Their cannot be a square which has a length of negative 27.Mathematicians were so opposed to the idea of negative numbers that there was not a single quadratic equation but 6 different versions all of which were arranged in a way so that the coefficients were positive,ensuring that negative solutions would not appear.The six equations being:

$ax^2 = bx$

$ax^2 = c$

$bx = c$

$ax^2 + bx = c$

$ax^2 + c = bx$

$ax^2 = bx + c$

This same approach was taken for the cubic. In the 11th century ,Omar Khayyam,a Persian mathematician,identified 19 different cubics equations,again keeping all coefficients positive.While this method allowed mathematicians to classify cubic equations,it fell short of providing a general solution.

Then in 1510, everything changed , at least for one man.Scipeno del Ferro found a reliable way to solve depressed cubics (a subset of cubic equations without the x^2 term).Such a breakthrough deserved celebrations.Instead ,del Ferro told no one.

Math duels

You see, back then mathematicians could challenge each other for their positions by sending each other a number of questions and the person who solved the most after an agreed period of time would be declared the winner of the “Math duel”.del Ferro knew that nobody knew how to solve these depressed cubics so by keeping it a secret he would keep his job shape.

He kept this secret till the very end only letting it slip on his deathbed to his student Antonio Maria Fior.Fior unlike his teacher boasts about his ability to solve the depressed cubics and challenges Niccolo Fontana Tartaglia a well known mathematician at the time.Tartaglia and Fior each send the other 30 questions.All of Fior's questions contain depressed cubic.The result 30 - 0 in favour of Tartaglia.Fior's boastfulness was his ultimate downfall as Tartaglia knowing that it was likely his questions would contain depressed cubics and knowing that their was a solution to depressed cubics had figured out a way to solve them before the Math duel.

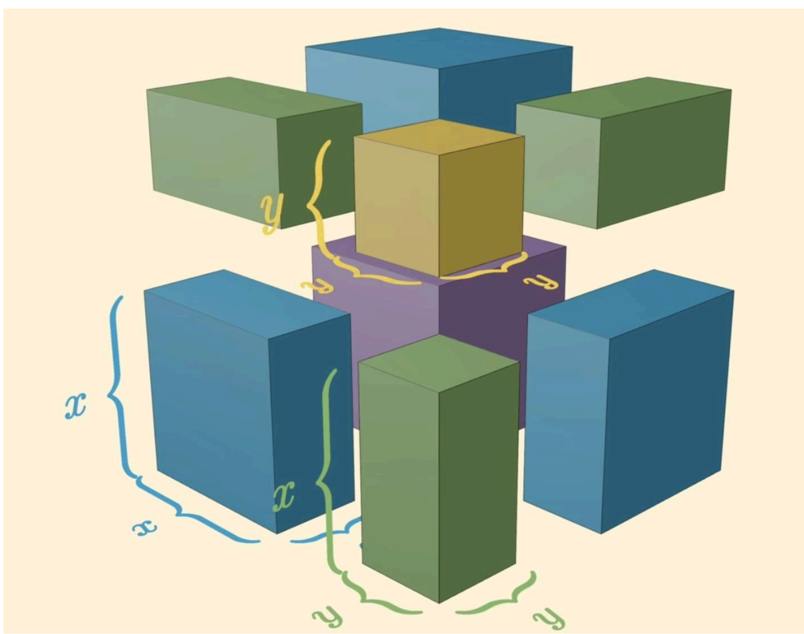
So what was Tartaglia’s method.Well he took the concept of completing the square and thinks of it 3 dimensionally.
Let's take the example of a question that came in the Math duel.

$x^3 + 9x = 26$

Now think of the x^3 as a cube of lengths x and $9x$ as a cuboid of volume $9x$.Adding these two shapes gives us a total volume of 26.

Next Tartaglia takes the original cube and increases its length by y so that $z = x + y$. With z being the new length.Now break this cube into 8 different shapes.

The original cube with length x , 3 cuboids of length x, x, y, 3 cuboids of length y, y, x ,and finally a cube with length y .



Source: Veritasium,Youtube,2022

Now Tartaglia formed a new cuboid (without using the two cubes).This new cuboid would have lengths $x, 3y, z$.Tartaglia then added y^2 to both sides of the equation giving us:

$x^3 + 9x + y^3 = 26 + y^3$

He then represents this new geometric shape as having a volume of $9x$.so we can equate the volume of cuboid to 9x giving us:

$$3yzx = 9x$$
$$3yz = 9$$
$$z = 3/y$$

Notice how the volume of the new cube is equal to the volume of two smaller cubes and the cuboid.Using this information:

$$x^3 + 9x + y^3 = z^3$$
$$z^3 = 26 + y^3$$

Solve the two equations we got simultaneously to get:

$$(3/y)^3 = 26 + y^3$$
$$27 = 26y^3 + y^6$$

This is a disguised quadratic.

$$(y^3)^2 + 26(y^3) - 27 = 0$$

Notice how this is the same quadratic we solved earlier!
Therefore,

$$y^3 = 1$$
$$y = 1$$

$$z = 3/y$$
$$z = 3$$

$$z = x + y$$
$$x = 2$$

Tartaglia became the second person to solve the depressed cubics and his victory made him famous with many mathematicians asking him how he was able to solve it.One of these mathematicians is Cardano who persistently sends letters to Tartaglia.Initially Tartaglia refuses but finally he agrees on the condition that Cardano swears not to tell anyone.Cardano agrees.

Cardano’s Discovery

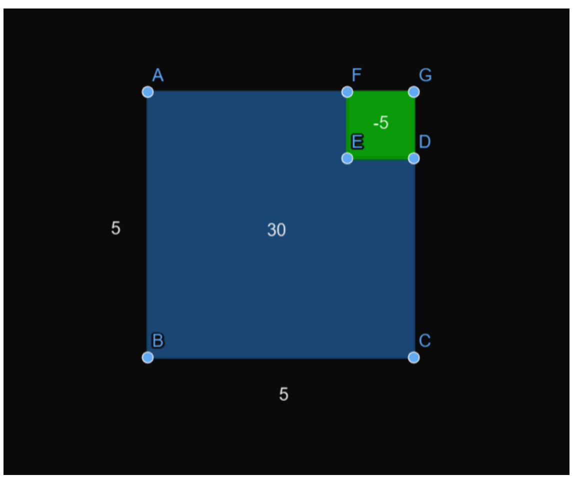
Cardano starts playing around with the equation and surprisingly comes up with a solution of the whole cubic equation including the x^2 term. Unlike his predecessors Cardano does not remain quiet and publishes his findings under a paper called Artis Magnae.

While writing Artis Magnae though Cardano came across equations that could not be easily solved.One such example is

$$x^3 = 15x + 4$$

This equation, much like the first equation, gives a solution containing the square root of negative numbers.However this wasn't the first time mathematicians came across square root of negative numbers and previously it was used as a way to identify that the equation has no solutions.

So the obvious conclusion is that their is no solutions, but simple trial and error shows that $x = 4$ is a solution. This stumped Cardano and he retraces his geometric derivation,exposing a fundamental breakdown.While the 3 dimensional cube slicing and rearrangements proceed without issue the final completing the square step leads to a geometric paradox: A region must have an area of 30 while simultaneously being a square of length 5.



In order to complete this square we need to add an area of -5.A concept utterly meaningless in classical geometry.

Cardano thinks this result is useless and moves on.However years later Mathematician Rafello Bombelli picks up where Cardano left off.

Bombelli's Method

Bombelli observes that the square root of negative numbers can neither be called positive or negative so he lets it be its own new number. He assumes Cardano's solution can be represented by an ordinary number and this new type of number.

Using Cardano's formula the solution is expressed as:

$$x = (2 + \sqrt{-121})^{1/3} + (2 - \sqrt{-121})^{1/3} = (2 + 11i)^{1/3} + (2 - 11i)^{1/3}$$

$$(2 + 11i)^{1/3} = a + bi$$
$$(2 - 11i)^{1/3} = a - bi$$
$$x = 2a$$

$$(a + bi)^3 = 2 + 11i$$
$$a^3 + 3a^2bi + 3ab^2i^2 + b^3i^3 = 2 + 11i \quad (note\ that\ i^2 = -1)$$
$$(a^3 - 3ab^2) + (3a^2b - b^3)i = 2 + 11i$$

$$a^3 - 3ab^2 = 2\ and\ 3a^2b - b^3 = 11$$
$$a = 2,\ b = 1$$

So:

$$(2 + i) + (2 - i) = 4$$

By adding both these solutions the square root term cancels out leaving the correct answer 4.

Cardano's method does give the correct answer but only if we abandon the geometric proof that built it in the first place.Negative numbers though physically meaningless are a necessary intermediate step.

Mathematicians later built upon these concepts ,Descartes who later dubbed these numbers imaginary numbers and Eulier who coined the notation i.

Reflection

Now this brings us back to our original question.

$x^2 + 1 = 0$

If solutions like $x = \pm i$ exists than the bigger question is where do they lie. To make sense of them mathematicians introduced the concept of complex plain.Unlike the Cartesian plane in the complex plain real numbers lie on the horizontal axis while imaginary numbers lie on the vertical axis. So a solution like $2 + 3i$
Would give the coordinates (2,3).This allows the imaginary numbers to be visualised and be integrated into algebra.

Conclusion

Where does this leave us now? Well imaginary numbers once dismissed as being “Imaginary” now prove essential to the physical world.For example at the quantum level the Schrodinger equation which governs quantum behaviour is almost entirely in terms of imaginary numbers.Furthermore imaginary numbers are used in Electricity where Current and voltage continuously oscillate making them impossible to monitor with ordinary numbers.Not only this but without imaginary numbers modern audio ,wifi,phone signals and much more would be impossible to process.

What began as a contradiction in solving equations, ultimately led us to discover the most powerful idea in mathematics.

Imaginary Numbers!