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Infinity and Paradoxes: How has Modern Mathematics resolved the Paradoxes that arise with the concept of Infinity?

1 Introduction

Due to my upbringing in a Christian household and the belief in an infinite God, I have always been fascinated by the concept of infinity; I often found it both paradoxical yet plausible. This fascination and curiosity stayed with me throughout my childhood, and I later encountered more advanced mathematics, such as superficial reckonings regarding infinity, which only deepened my curiosity. Unfortunately, even at A-level, infinity is not explored in depth; although it appears in topics such as limits, the underlying reasoning is often omitted. I, however, am a firm believer that understanding why mathematical concepts work is essential for deeper comprehension of the subject. Therefore, I decided to take it upon myself to dive deeper into the iceberg that is the concept of infinity, beyond the curriculum, and following this introduction will be the results of some of my findings.

2 'Defining' Infinity

The English word 'infinity' derives from the Latin *infinitas,* which literally means not finite or not defined, and to explore these paradoxes, we must first consider what is meant by infinity - or at the very least try to, considering that there is no single universal definition. This is largely because mathematicians have found it functionally useless to try and fit infinity within the parameters of a single definition, but also due to the fact that the concept's complexity has yielded itself to a plethora of definitions that are all entirely dependent on the specific mathematical context, i.e. the way one would define infinity when using it in set theory (actual infinity) is completely different to the way one would define it when it appears as a limit in calculus (potential infinity). For the purposes of this essay, the definitions of both actual and potential infinity are required. Hence, actual infinity is defined as:

> On the one hand a multiplicity can be such that the assumption of ***all*** of its elements 'are together' leads to a contradiction, so that it is impossible to conceive of the multiplicity as a unity, as 'one finished thing'. Such multiplicities I call ***absolutely infinite.*** [1]

[1]Richard Arthur, Quoted in **Leibniz and Cantor on the Actual Infinite,** pp.6-7, 2001

This definition is fundamentally oppositional to that of potential infinity, which is as follows: “there is always another and another to be taken. And the thing taken will always be finite, but always different” [2]

Hence, with these definitions established, let us explore two paradoxes that arise with the concept of actual infinity, and one that arises as a result of blurring the lines between actual and potential infinity.

3 Hilbert’s Paradox of the Grand Hotel

Proposed by German mathematician and philosopher David Hilbert (1862 - 1943) in the 1920s, Hilbert’s Paradox of the Grand Hotel is widely considered the most popular thought experiment used to illustrate the counterintuitive properties of actual infinity, more specifically countable sets, and acts as a practical demonstration of Georg Cantor’s mathematical theory of transfinite numbers. For some context, Hilbert originally introduced it in a lecture of January 1924, but without publishing it, and so its inception remained largely unknown until it was popularised by George Gamow’s 1947 book *One Two Three… Infinity*, and only gained significant traction around three decades later in the 70s.[3] That being said, the paradox revolves around the premise that an infinite, finished set of rooms already exists in its entirety, rather than a process of continually adding new rooms, allowing for the manipulation of transfinite sets. It goes as follows: Consider a hotel with infinitely many rooms, all occupied by a maximum of one guest, numbered using the set of natural numbers, $\mathbb{N} = \{1, 2, 3, \ldots\}$. What happens when more people want to check in? The hotel is infinitely full, so surely it must be impossible to accommodate any new guests? Well, this is where the paradox arises, and the rest of this section will demonstrate how such scenarios within Hilbert’s Hotel can be resolved. Suppose one customer enters the hotel despite there theoretically being no vacancies. A novice to the concept of infinity would likely turn the customer away under the assumption that the hotel has no room for them. You, on the other hand, are familiar with the concept and so tell everyone in the hotel to move to room $n + 1$, i.e. the occupant of room 1 moves down to room 2, the occupant of room 2 moves down to room 3, and this process repeats such that each room is occupied except room 1.

[2] Øystein Linnebo and Stewart Shapiro, Quoted in **Actual and Potential Infinity**, p.167, 2019
[3] Halge Kragh, **The true (?) Story of Hilbert’s Infinite Hotel,** 2014

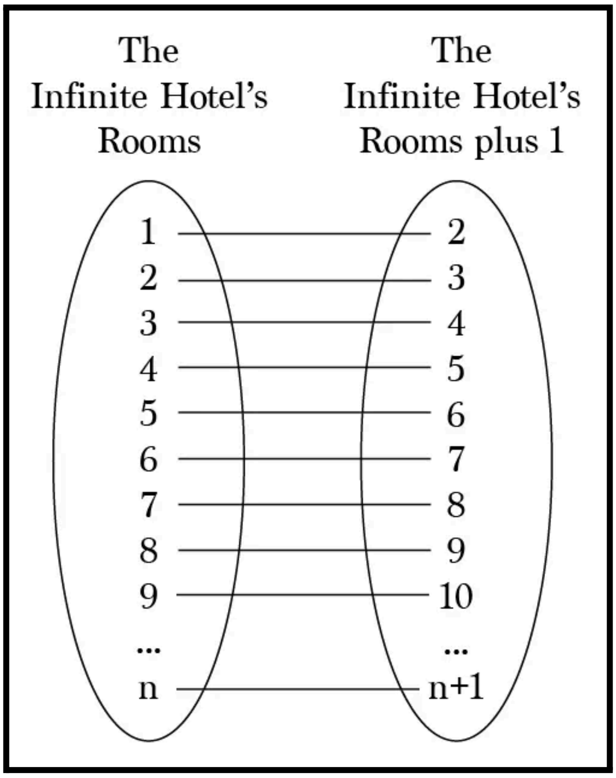

Figure 1: Hilbert's Infinite Hotel Paradox, Brett Berry, 2017

This consequently allows the new guest to be granted the now vacant room 1. Suppose now an infinitely-long bus containing an infinite number of passengers appears, what will you do now? A natural approach would be to do what you did previously and tell everyone to move down one room, but because we are adding infinity to infinity ($\infty + \infty$) and not a finite value to infinity $(finite + \infty)$, this conundrum must be approached from a different angle as it would not be plausible to tell everyone to move infinitely many rooms down. Instead, guests are reassigned such that everyone moves to room $2n$ (i.e. the occupant of room 1 moves to room 2, the occupant of room 2 moves to room 4, etc.) so that only even-numbered rooms are occupied *(because any integer that has a factor of 2, i.e.* $2n$*, is even),* thus leaving all odd-numbered rooms vacant. Since there is an infinite number of even numbers as well as an infinite number of odd numbers, everyone from the infinitely-long bus can be granted an odd-numbered room, hence solving this seemingly unsolvable problem. However, imagine now an infinite number of buses in the likeness of the one from the previous scenario appear, what can be done to make room this time? Initially, your first thought process is to do the same trick as before, and while it is a good starting point, you want to start thinking more intuitively. So, you move one of the many passengers into room 1, and then you number the buses using the set of odd prime numbers, $\mathbb{P} = \{x \in Prime : x > 2\}$ such that Bus One is assigned the number 3, Bus Two is assigned the number 5, so on and so forth until infinity. Following this, you then tell the passengers of each bus to move into room $\mathbb{P}^n$, i.e. Bus 1 passengers: seat n goes to room 3^n (Bus 1, Seat 1 $\rightarrow 3^1$, B1S2 $\rightarrow 3^2$, etc.), Bus 2 passengers: seat n goes to room 5^n, Bus *b* passengers: seat n goes to room P^n_{b+1}, where P_{b+1} is the $(b\ +\ 1)th$ prime number. Because of the Fundamental Theorem of Arithmetic, which states that every integer greater than 1 can be factored uniquely into a product of primes [4], every passenger will be given a unique number and thus no two passengers will ever be assigned the same room.

Evidently, modern mathematics has proven that despite being 'full', the hotel can still accommodate an endless number of new guests simply by rearranging existing occupants, essentially illustrating that the set of natural numbers is stretchable. This demonstrates that countably infinite sets all have the same cardinality, aleph null, which at first glance seems impossible as you would assume that, in the case of Hilbert's Infinite Hotel, the infinity contained by the buses and passengers should be greater than the infinity the hotel can hold, but since the infinite number of buses can be represented by the set of prime numbers (excluding 2), and the number of rooms is the set of natural numbers, it can now effectively be paired one-to-one, even though prime numbers are only a subset of natural numbers. This then is how Hilbert's Paradox of the Grand Hotel deals with the logical, counterintuitive consequences of actual infinity, which in this case is a completed, existing, set of infinite rooms and guests.

4 Zeno's Paradoxes of Motion

Zeno of Elea (490 - 430 BC) was a pre-Socratic Greek philosopher known for designing a series of paradoxes that argued change and motion are merely illusions and do not exist in reality. Such paradoxes include the dichotomy paradox and the paradox of Achilles and the Tortoise, both of which I will be exploring further due to their intertwined nature. Both paradoxes essentially operate on the premise that for motion to occur, i.e. for a set distance to be covered, S, one must first cover at least half of S, $\frac{1}{2}S$, but in order to cover $\frac{1}{2}S$ one must first cover at least a half of that, or in other words $\frac{1}{4}S$. As you can likely imagine, this sequence goes to infinity, and the sum of which would be calculated using the sum to infinity formula for a geometric series. A more in-depth description of the paradox of Achilles and the Tortoise goes as follows:

> Let us assume that there is such a thing as real motion. Now if there is such a thing as real motion, it is certainly an impossibility that Achilles cannot catch a tortoise, when Achilles is running twice as fast as the tortoise (let us say) and the distance between the two is finite. But it appears on such-and-such an analysis… that Achilles cannot catch this tortoise. Therefore, by *modus tollens*, there can be no such thing as real motion. [5]

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[4] Bruce Ikenaga, University of Houston, **The Fundamental Theorem of Arithmetic**, p.1, 2008

[5] John O. Nelson, **Zeno's Paradoxes on Motion,** p.486, 1963

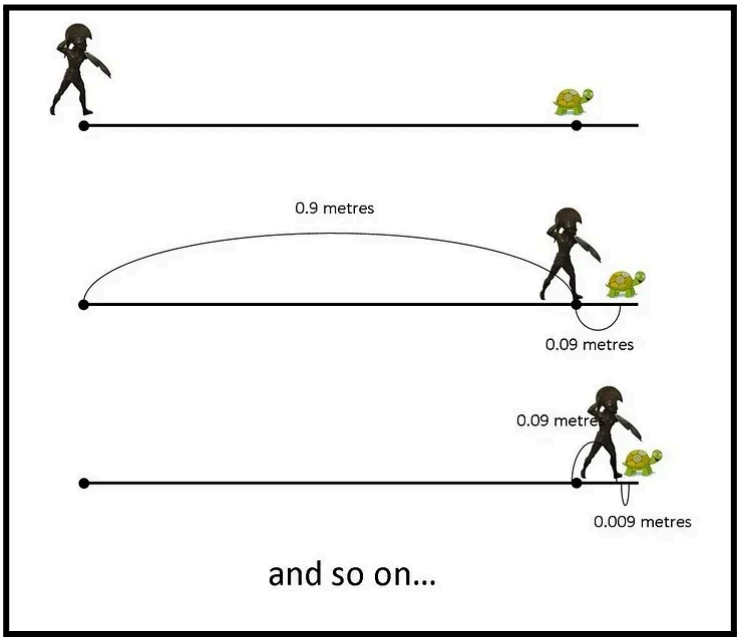

Figure 2: On Zeno's Paradoxes, Suhasnalla, 2023

Zeno proposes that there will always be an infinitesimally small distance between Achilles and the tortoise, thus motion is an illusion and not real. Although this contradicts observable reality, as any observer of the real world will be able to give you a posteriori evidence that motion is undoubtedly real, but a formal resolution to the paradox was historically lacking. However, due to the evolution of modern mathematics, there are now many resolutions to the paradoxes, and one of them is as follows:

let $S = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots$

$\Rightarrow \frac{1}{2}S = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$

$\Rightarrow S - \frac{1}{2}S = \frac{1}{2}S = \frac{1}{2} \Rightarrow S = 1$

Alternatively, if we were to write *S* as a sequence, it would be geometric and the common ratio, *r*, would be $\frac{1}{2}$, and since $\left|\frac{1}{2}\right| < |1|$, the series is convergent and therefore has a finite sum. That finite sum can be calculated using the formula $S_{\infty} = \frac{a}{1-r} \Rightarrow S_{\infty} = \frac{\frac{1}{2}}{1-\frac{1}{2}} = 1.$

Therefore, Zeno's intuition about being able to split a process into infinitely many parts, albeit correct, did not justify his flawed conclusion of motion subsequently being unreal, as mathematics is now able to prove that an infinite series can still have a finite sum. Zeno's paradox is ultimately illegitimate, and "the faulty logic in Zeno's argument is often seen to be the assumption that the sum of an infinite number of numbers is always infinite, when in fact, an infinite sum… can be mathematically shown to be equal to a finite number".[6] This means despite the potential to divide a length by two an infinite number of times, the length itself is still finite, and said length can be covered in a finite amount of time, hence motion is possible. In the case of Achilles and the Tortoise, although Achilles must reach an infinite number of points where the Tortoise previously was, those points get progressively and infinitely close, allowing him to pass the tortoise in a finite total time. [7]

5 Conclusion

Paradoxes arising from infinity are not flaws in mathematics, but reflections of the limitations of human intuition. As demonstrated through Hilbert's Hotel and Zeno's paradoxes, apparent contradictions emerge when finite reasoning is applied to infinite processes. Modern mathematics resolve these issues through rigorous definitions and formal frameworks, showing that infinity can be treated consistently despite its counterintuitive nature. Although this essay has explored only a small part of the topic, it highlights how mathematical structure allows seemingly paradoxical ideas to be understood coherently.

[6] Peter Lynds, **Zeno's Paradoxes: A Timely Solution,** p.4, 2003
[7]Ibid, p.4