Infinity seems impossible - so how can it be useful?

Introduction: The Problem with "Full"

In our ordinary lives, the word "*full*" is usually a dead end. If a bus has forty seats and forty people are sitting in them, the bus is clearly full and has no space available. If a hotel has one hundred rooms and one hundred keys are assigned to guests, the hotel is obviously full, and no more guests can be accommodated.

We can see that our brains have evolved to only understand finite capacities, but when infinity is introduced, this logic completely changes. In an attempt to help people understand how something "*entirely full*" can still have room for more, the mathematician **David Hilbert** created a paradox known as **Hilbert's Hotel**.

The Single Guest

Imagine there is a hotel with infinitely many rooms, each corresponding to one person and to a natural number (1, 2, 3, 4…). This type of infinity is known as a *countable infinity,* meaning each room can be matched with a natural number. Now imagine you are the receptionist, and every single room is occupied.

A new guest walks in and requests a room. Your initial intuition might be to put the guest into the "*last*" room, but since we are dealing with infinity, there isn't actually a final room, and even if there was, it would be occupied. So, instead of looking for a vacancy, you create one. You ask the guest in Room 1 to move to Room 2, the guest in Room 2 to move to Room 3, and so on.

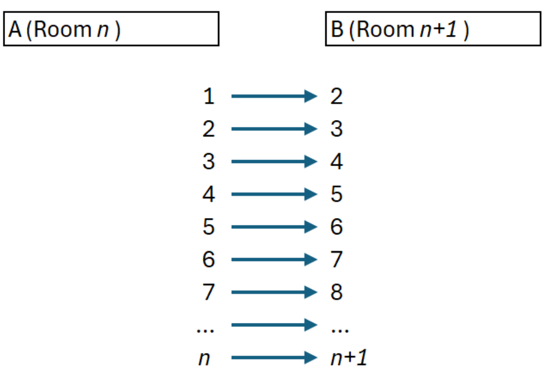

Figure 1: Guests in Hilbert's Hotel moving from room n to room $n+1$. *Source: Author's own work.*

As shown in **Figure 1**, we have shifted each guest from Room *n* to Room *n*+1, so an infinite number of people have simultaneously moved down a room, leaving Room 1 vacant for the new guest. With this method, any finite number of new guests can be accommodated. Perhaps we let *k* be the number of new guests, we would arrange the current guests to move from their room, *n*, to Room *n*+*k*. This proves a fundamental set theory rule: adding a finite number to an infinite set doesn't make it larger.

In mathematical terms, this is described as:

$$\aleph_0 + k = \aleph_0$$

However, this concept feels counterintuitive because, with finite numbers, 10 + 1 is certainly greater than 10. However, in terms of infinity, it depends on whether you can pair each guest with a room number. Since you can still fit each guest into the same infinite list of rooms, the "size" remains the same. Therefore, you haven't created a larger infinity but instead rearranged the one you already had.

The Infinite Bus

The corresponding night, a new challenge arises. An infinitely long bus pulls up outside, containing an infinite number of new people. To accommodate them, you don't just shift the guests by one, but simply use basic multiplication. By ordering each current guest to move from their current room, *n*, to Room 2*n*, every odd-numbered room now becomes available.

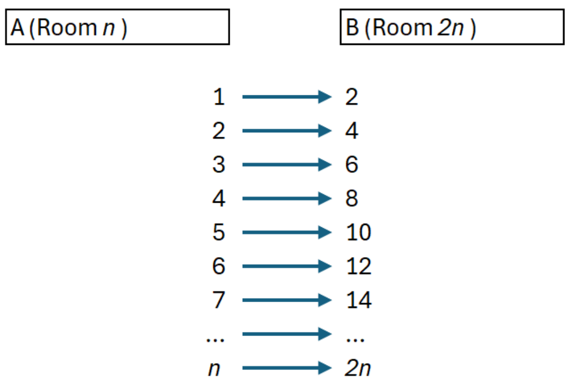

Figure 2: Infinite guests in Hilbert's Hotel moving from room *n* to room *2n*. *Source: Author's own work.*

Following the logic in **Figure 2**, the guest in Room 1 moves to Room 2, the guest in Room 2 moves to Room 4, the guest in Room 3 moves to Room 6, and so on. As there's an infinite number of odd-numbered rooms available, the entire bus can now check in. This proves $\aleph_0 + \aleph_0 = \aleph_0$.

The Infinite Buses

It gets stranger, however. An infinitely long boat arrives, carrying an infinite number of buses, each carrying an infinite number of people. To solve this requires a more advanced system, so no two people overlap. We now turn to **Prime Number Factorisation**.

A (Seat n and Coach k)	B (Room $2^n 3^k$)
Seat 1 Coach 1	6
Seat 2 Coach 1	12
Seat 3 Coach 1	24
Seat 4 Coach 1	48
Seat 1 Coach 2	18
Seat 3 Coach 2	72
Seat 3 Coach 3	216
...	...
Seat n Coach k	$2^n 3^k$

Figure 3: Seat and Coach number corresponding to room number in Hilbert's hotel. *Source: Author's own work.*

As seen in **Figure 3**, we assign each "category" of guest a unique prime number. Existing guests in Room *r* are moved to Room 2^r. For new arrivals, we assign each person in Seat *n* and Coach *k* to the room $2^n 3^k$, uniquely identifying each guest. For example, a guest in Seat 3 and Coach 2 would be assigned to Room $2^3 * 3^2 = 72$.

With this method, all guests are accommodated without overlap, because every number has its own unique prime factorisation. Furthermore, all other prime factors other than *2* and *3* are left unused, so it can be used to model for infinite boats ($2^n 3^k 5^f$), infinite dates ($2^n 3^k 5^f 7^d$), and so on, as we are still working within a countable infinity ($\aleph_0$).

When Infinity Breaks

Finally, an infinitely long bus arrives with guests whose names are infinite strings of 1s and 0s. For example, one guest might be called 101010101010…, another could be 010111111111…, and so on, forever. You might try to list them to assign each to a room, but this quickly proves impossible.

We now see the leap from a *countable infinity* to an *uncountable infinity*.

A (Room number)		B (Guest's name)
1	→	11111111111...
2	→	10101010101...
3	→	10111111111...
4	→	10000000001...
5	→	00000000011...
6	→	00000011110...
7	→	11101101011...
...	→	...

B (Guest's name)		C(Guest's start of name without room)
11111111111...	→	0
10101010101...	→	01
10111111111...	→	010
10000000001...	→	0101
00000000011...	→	01011
00000011110...	→	010111
11101101011...	→	0101111
...	→	...

Figures 4 and **5:** Cantor's Diagonal Argument. Source: *Author's own work.*

As described in **Figures 4** and **5**, regardless of you attempt to list these guests, there will always be one who is not accommodated. By using *Cantor's Diagonal Argument*, we can construct a name of digits that is guaranteed not to be assigned to a room.

If we take the first digit of the first name, the second digit of the second name, and so on, then flip these digits, we create a name that is different from every listed guest accommodated in **Figure 4** for at least one digit. As this bus contains every combination of names containing these digits, this name cannot already be assigned to a room.

This proves that infinities can be larger than others.

Conclusion: The utility

So, is Hilbert's hotel just a confusing brainteaser? Even though it is impossible, the logic behind it is extremely useful. It helps mathematicians understand that not all infinities are the same size. Some infinities can be rearranged to make room for more guests, but when larger infinities are introduced, the hotel fails.

Even infinity can be too small (if you know the maths behind it).

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