Infinity is Weird – A Short History

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I Introduction

What is infinity? The short answer is 'complicated' and the long answer is the rest of this essay. A controversial mathematical and philosophical concept since antiquity, infinity has only become more counterintuitive as we've come to understand it more. This essay will lay out a linear, if selective, timeline of discoveries from the earliest understanding of infinity to possibly the most infamous example of infinity defying all intuition – the Banach-Tarski Paradox – in a showcase of how this seemingly simple idea has confused the greatest human minds for all of time.

II Infinity in Antiquity

Although a rigorous definition of infinity would not be formalised for some millennia afterwards, the Ancient Greeks dealt often with the implications of philosophical infinity; Most famously, Zeno of Alea proposed his "Paradoxes of Motion" in the fifth century BCE. Zeno asks us to consider a footrace, between the great hero Achilles and a tortoise, wherein the tortoise has been given a headstart, let's say 100 meters – He proposes that Achilles, despite being faster than the tortoise, can never overtake it as anytime Achilles catches up to the tortoise, it will have moved just a little further.

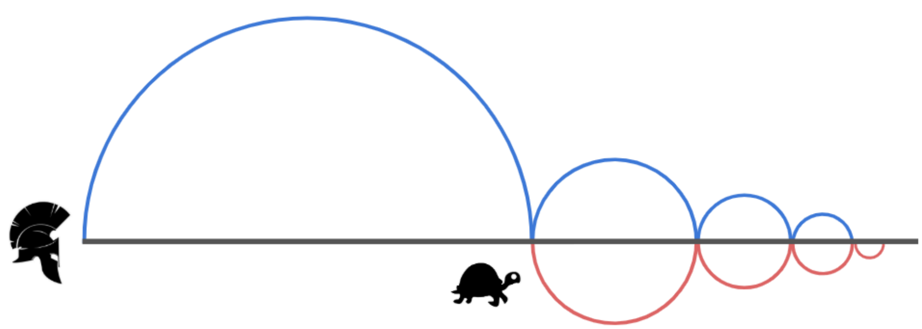

Fig. 2.1

Although Zeno never did so (owing to his lifetime of 460 - 430 BCE), we can represent this paradox mathematically as a proof by induction;

> Let the speed of the tortoise be denoted V_T and let the speed of Achilles be denoted V_A
> Let their respective distances at time *t* seconds be denoted $S_T(t)$ and $S_A(t)$
>
> Now consider the base case at t_1 s.t. $S_T(0) = V_A \cdot t_1$ (i.e. Achilles makes it to the Tortoise's starting position after t_1 seconds);
>
> $$S_T(t_1) = 100 + V_T \cdot t_1$$
> $$V_T, t_1 > 0 \Rightarrow 100 + V_T \cdot t_1 > 100 \stackrel{\text{def}}{=} S_A(t_1)$$
> $$\Rightarrow S_T(t_1) > S_A(t_1)$$
>
> Now consider the inductive case;
>
> Assuming $S_T(t_n) > S_A(t_n)$,
> Define t_{n+1} $s.t.$ $S_T(t_n) = S_A(t_n) + V_A \cdot (t_{n+1} - t_n)$,
> $$\Rightarrow S_T(t_n) = S_A(t_{n+1})$$
> $$S_T(t_{n+1}) = S_T(t_n) + V_T \cdot (t_{n+1} - t_n)$$
> $$t_{n+1} > t_n \Rightarrow (t_{n+1} - t_n) > 0$$
> $$(t_{n+1} - t_n), V_T > 0 \Rightarrow S_T(t_n) + V_T \cdot (t_{n+1} - t_n) > S_T(t_n) = S_A(t_{n+1})$$
>
> $$\therefore S_A(t_{n+1}) < S_T(t_{n+1})$$
>
> Thus, via the principle of induction we can conclude that;
>
> $$\forall\, t_n, t_{n+1}\ s.t.\ S_T(t_n) = S_A(t_n) + V_A \cdot (t_{n+1} - t_n), n \in \mathbb{N},$$
> $$S_A(t_{n+1}) < S_T(t_{n+1})$$
>
> I.e. we can conclude that at every time at which Achilles has caught up to the Tortoise, the Tortoise remains ahead.

This is one of three 'Paradoxes of Motion' proposed by Zeno, and from these he concluded that motion was a logical impossibility, and yet motion clearly exists, as proven in one case by Diogenes the Cynic who, upon hearing Zeno propose his paradoxes, simply walked away in silence – *Solvitur ambulando*, it is solved by walking. Over 2000 years later, French mathematician and pioneer of Analysis, Louis-Augustine Cauchy, would provide a rigorous proof that certain geometric series would converge, inadvertently providing a method to find the point at which Achilles overtakes the tortoise.

> Consider that;
>
> $$\Delta S_T(t_n) = \frac{V_T}{V_A} \cdot \Delta S_A(t_n),\ \Delta S_A(t_n) = \Delta S_T(t_{n-1})$$
> $$\Rightarrow \Delta S_T(t_n) = \frac{V_T}{V_A} \cdot \Delta S_T(t_{n-1}) = \left(\frac{V_T}{V_A}\right)^2 \cdot \Delta S_T(t_{n-2}) = \left(\frac{V_T}{V_A}\right)^n \cdot \Delta S_T(0)$$
>
> Using Cauchy's 1821 formula;
>
> $$\sum_{n=0}^{\infty} ax^n = \frac{a}{1-x}\ ,\ x \in (0, 1)$$
> $$\sum_{n=0}^{\infty} \Delta S_T(t_n) = \sum_{n=0}^{\infty} \left(\frac{V_T}{V_A}\right)^n \cdot \Delta S_T(t_0) = \frac{\Delta S_T(t_0)}{1-\left(\frac{V_T}{V_A}\right)}$$
>
> This is the distance at which Achilles overtakes the tortoise.

The philosopher James F. Thomson called this process, of performing an infinite number of tasks in a finite space of time, a Supertask.

III Counting Infinity

Now moving into the modern era, we move on to another mathematician, set theorist Georg Cantor. Cantor's work focused on two ideas, counting infinity and ordering infinity, let us focus on the first of these. We'll begin with a simple question, how many natural numbers are there? The answer is obvious, there's an infinite amount – this can be proven by an incredibly trivial contradiction;

> Assume $\exists\ N$ which is the largest natural number;
> Given that $a, b \in \mathbb{N} \Rightarrow (a + b) \in \mathbb{N}$;
>
> $$(N + 1) \in \mathbb{N}$$
> $$N + 1 > N$$
> $$\perp$$
>
> Hence, there is no largest natural number, thus there are infinitely many natural numbers

Although there is no normal number to describe the size of $\mathbb{N}$, Cantor elected still to define the amount of natural numbers, even if infinite, giving it the symbol $\aleph_0$ (Aleph Null). Infinite sets such as $\mathbb{N}$ do not behave in the same way as finite sets, to demonstrate this we first need to talk about cardinality – The cardinality of a set, denoted $|S|$ describes the number of elements in the set.

> For example,
>
> Let $S \overset{\text{def}}{=} \{1, 2, 3, 4, 5, 6, 7\}$
> Then $|S| = 7$

If we want to compare the cardinality of two sets, we look for a bijection. A bijection is a function where no information is lost – i.e. a reversible function wherein every unique input has its own unique output. Note for later that this can formally be expressed as two conditions;

> For $f: A \rightarrow B$
> Injectivity;
>
> $a, b \in A$
> $f(a) = f(b) \Rightarrow a = b$
>
> And surjectivity;
>
> $\forall b \in B, \exists\, a \in A\ s.t.\ f(a) = b$

If there is a bijection between two sets, they have the same cardinality. While this is perfectly intuitive for a finite set;

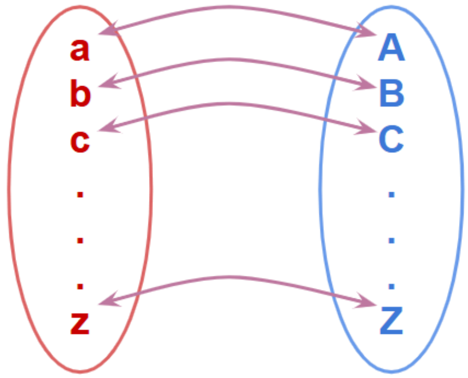

Fig. 3.1

The intuition begins to break down as we look at infinite sets. Consider for example, the question "Are there more integers or even numbers?" The answer seems simple; given that the even numbers form a proper subset of the integers, we might assume that the set of integers is larger. This intuition would be perfectly sound for finite sets, however for infinite sets, it breaks down.

> Let $E \stackrel{\text{def}}{=} \{2k | k \in \mathbb{Z}\}$,
> Consider $f: x \mapsto 2x$
>
> $f(a) = f(b)$
> $\Rightarrow 2a = 2b$
> $\Rightarrow a = b$
> Hence $f: x \mapsto 2x$ is injective
>
> $\exists\, c \in \mathbb{R} \Rightarrow \exists\, 2c \in \mathbb{R}$
> $2c = f(c)$
> $\Rightarrow \forall\, d \in \mathbb{R}\, \exists\, c \in \mathbb{R}\; s.t.\; f(c) = d$
> Hence $f: x \mapsto 2x$ is surjective
> Hence $f: x \mapsto 2x$ is bijective
>
> Consider that;
>
> $\mathbb{Z} \equiv \{k | k \in \mathbb{Z}\}$
> $\Rightarrow f: \mathbb{Z} \to E$
>
> Given f is bijective, this implies that;
>
> $|\mathbb{Z}| = |E|$
> I.e. there are exactly as many even numbers as integers

This revelation may seem either entirely reasonable or unreasonable depending on how one approaches it. On one hand it seems silly that a proper subset should be identically sized to its superset, but on the other hand both are infinite – and surely all infinities are the same? In case that wasn't on-the-nose enough, not all infinities are in fact the same; Cantor showed this, not first but most aptly, in his so-called *Diagonal Argument*.

Cantor proposes to assume we can bijectively pair the Natural and Real numbers as such;

1	0.232683920419
2	1.353567335733
3	0.947230038104
4	2.401984735450
5	0.193435893629
6	1.191900000000
…	…

He then produces a new real number by changing the first digit of the first number, second of the second, and so on – thereby producing a real number which is not included in the original mapping.

1	0.232683920419…
2	1.353567335733…
3	0.947230038104…
4	2.401984735450…
5	0.193435893629…
6	1.191900000000…
…	…
R	1.45251…

Notably, this can be repeated even after incorporating the new real number into our mapping, thus proving that we can never produce a complete bijective mapping $\mathbb{N} \to \mathbb{R}$ and therefore that these sets, despite both being infinite, are of different cardinalities.

IV Cantor's Quest to Well-Order the Real Numbers

Georg Cantor's most lasting work was not his "counting" of infinite sets however, but his ordering of infinite sets. Cantor defined a *Well Ordering* with two conditions;

> Condition I.
>
> $\exists\, a \in S\, s.t. \forall b \in S \backslash a,\ i_a < i_b$
> Where i_a denotes the index of a
> I.e. there is a least element in S
>
> Condition II.
>
> $\forall S_0 \subset S, S_0 \neq \emptyset \exists\, a \in S_0\, s.t. \forall b \in S_0 \backslash a,\ i_a < i_b$
> I.e. there is a least element in all non-empty subsets of S

And with those two seemingly simple conditions, Cantor set himself on the course for a work that would take up his entire life and even institutionalise him at one point – the quest to well-order the real numbers. One of the first issues we run up against is "countability" – namely, while the Natural numbers are countable, 1, 2, 3, etc. the Real numbers are not; for any given Real number, there is no "next" number (in order of magnitude).

> $\forall a, b \in \mathbb{R} \exists \frac{a+b}{2} \in \mathbb{R}$
> $a < b$
> $\Rightarrow a < \frac{a+b}{2} < b$
> I.e. there will always be another real number in between any two real numbers

For this reason it isn't possible to identify a distinct well-ordering of the real numbers, but Cantor's endeavour was only to prove that one exists, not to identify it. An additional problem is the process of indexing the set of Real numbers; as we showed earlier there are more Real numbers than Natural numbers and so it is not possible to index the Real numbers with Naturals alone. Luckily, indices do not necessarily need to be cardinal numbers (i.e. quantities), only ordinals (i.e. ordered), and so we can define new ordinal numbers as such;

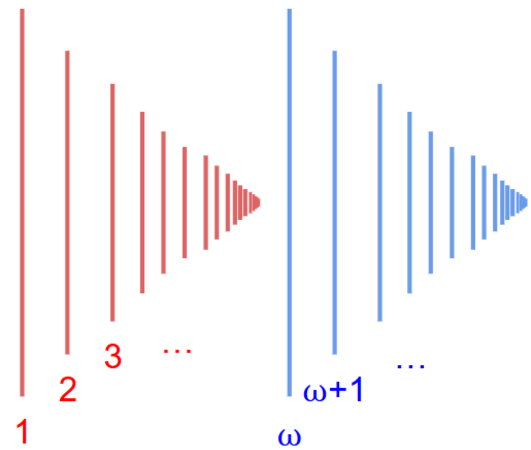

Fig. 4.1

Here, we define omega to be the least ordinal greater than all natural numbers, allowing us to index sets of greater cardinality than $\aleph_0$. This process is called transfinite induction.

The final piece of the puzzle would elude Cantor for 9 years, and in fact it was not Cantor who ultimately discovered it but a young(ish) German logician, Ernst Zermelo. The final piece in question was the Axiom of Choice.

Axioms are foundational assumptions, the basic principles of mathematics that we build all other proofs on top of. The Axiom of Choice states that it is always possible to uniquely select one element from every non-empty set in an infinite collection of sets. This is logically equivalent to saying that the Cartesian Product (The set of all combinations featuring one element of each set) of any collection of non-empty sets is itself non-empty, or more formally;

$$S_i \neq \emptyset \; \forall \, i \in I \Rightarrow \prod_{i \in I} S_i \neq \emptyset$$

With this Axiom, Zermelo proved in 1904 what Cantor had conjectured (to much ridicule) in 1895, that all sets, including the Real numbers, could be well-ordered;

> Let the set we are well-ordering be A and let f be a choice function on A as allowed by the axiom of choice;
>
> For an ordinal i (determined via transfinite induction) use the choice function to define an element $a_i \in A$;
>
> $$a_i = f(A \backslash \{a_n | n < i\})$$
>
> Once repeated for all ordinals, this process will enumerate every element in A, thus creating a well-ordering.

And with that Zermelo and Cantor had proven that a well-ordering of the Real numbers both necessarily existed and was simultaneously impossible to ever discover.

This proof, and even the axiom itself, may seem trivial – this owes in part to my simplification of the logic but also in part to the fact that it was somewhat. As many mathematicians noted after Zermelo published his proof, the Axiom of Choice had been implicitly used in many proofs for centuries, but it was never formalised until Zermelo; Once it had been however, it opened the door to a whole host of completely counterintuitive results, mostly infamously the Banach-Tarski Paradox.

V The Banach-Tarski Paradox

Despite the name, the Banach-Tarski Paradox is not a true paradox but a veridical paradox – while, as we'll see, it defies all our geometric intuition, it is entirely true and not in any way self-contradictory. The proof of the Banach-Tarski Paradox is needlessly complicated and would require introducing group theory, which I haven't the remaining room in the word count to do – As such, this section will be slightly more conceptual than rigorously mathematical.

Consider, as Ian Stewart does in his 1996 book *From Here to Infinity*, a dictionary containing every possible word using the latin alphabet – our infinite dictionary would begin with A, followed by AA, AAA, AAAA et cetera, eventually followed by AB, ABAA, ABAAA et cetera. Stewart calls this a *Hyperwebster* and notes how the Hyperwebster contains itself. In fact not only does it contain itself, it does so infinitely many times.

> Consider any string in the Hyperwebster;
>
> *ABCDEFGHIJKLMNOPQRSTUVWXYZ*
>
> Now note that, by definition of the Hyperwebster, there must exist another string, identical to our select string, only with a preceding character;
>
> *[...]ABCDEFGHIJKLMNOPQRSTUVWXYZ*
>
> It follows then, if we take the 'chapter' of the Hyperwebster beginning with our preceding character [...], then remove said character, we are left with the original Hyperwebster. This process can be repeated as many times as one wishes.

With this intuition in mind, the ability of sets to be proper subsets of themselves, we can apply a similar idea with directions. If we define an infinite list of 4 ordered directions LRUDLR… etc in the same manner as the hyperwebster, we can remove any initial step from a "quarter" of the set and, in combination with the "opposite" quarter, produce new sets identical to the original;

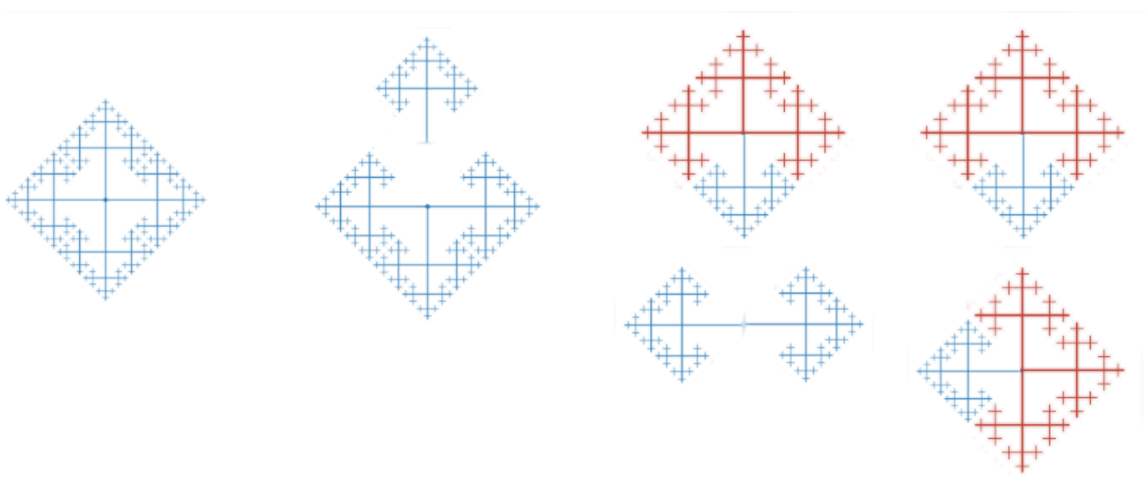

Fig. 4.1

Note that, although the diagram is scaled for this visualisation, in reality the scale has no bearing on the actual mathematics behind the paradox.

We can apply this finally to a sphere; using the axiom of choice we select infinitely many starting points, from which we apply the previous process of creating infinite sets of dimensions and separating them, resulting eventually in two spheres.

As mentioned, this is only a veridical paradox; it is mathematically sound and only defies our intuition because it highlights the chasm between two branches of mathematics; set theory and Euclidean geometry.

VI Conclusion

To conclude, for as long as humanity has considered it, mathematically or philosophically, infinity has defied all our expectations. This essay only scratches the surface; infinity similarly snubs our intuition in the cases of Gabriel's Horn, Hilbert's Hotel, and many other humbly named thought experiments. Ultimately while it may seem questionably relevant – indeed many philosophers deny its existence outright – the infinite can be a window into the most brilliant and abstract areas of mathematics.

Note: I do not know if these essays are judged by Dr. Crawford alone or by other people, so I will reiterate something mentioned in an email to Dr. Crawford. Depending on the program used, this essay may be considered to have a word count of ~2600 – this is in error as most word-counting programs count individual characters of notation as entire sentences. In reality the body of this essay, this note notwithstanding, has exactly 1999 words as stipulated in the rules of the competition. Finally, to cover all my bases, I am perfectly aware of the grammatical applications of, and frequently employ, em dashes – no generative AI was used at any stage of this essay, all research, writing, and graphics were entirely my own.