Learning to Trust Your Calculator: An Exploration of How Calculators Compute Non-Polynomial Functions

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1 Introduction

Try this: take your calculator and type in e^{π} and watch what happens. You get a neat decimal $23.14069...$ appearing almost instantly on your display, with no indication anything particularly complicated has taken place. Now try $ln\ 2$: again, the calculator will confidently serve us an answer $(0.693147...)$, but do you really understand *where* that number came from - or how it could possibly be computed at all?

For most high-school students encountering these sorts of functions for the first time, their calculator quickly becomes a magical black box: we press buttons, receive answers, and simply move on without stopping to question the process behind them. Yet, this blind trust in our calculators is not entirely justified.

Functions like e^{x}, $ln\ x$, and $sin\ x$ fundamentally *transcend* algebra. They are called transcendental functions, because they cannot be expressed as a finite combination of algebraic operations and are thus impossible to calculate analytically for general inputs.

Contrarily, a calculator, under all the layers of abstraction, is fundamentally limited to performing these basic arithmetic operations: addition, subtraction, multiplication and division. So, then, how do our calculators evaluate functions that go beyond these operations?

This essay answers that question —and reveals why the calculators' deserve our trust.

2 The Power Series for e^x

Let us begin by understanding how calculators find e^{x}. One method used to evaluate non-polynomial functions is by approximating them via Taylor's Theorem.

A Talor series is used to express a smooth function as an infinite sum of polynomial terms centred around a point a. If a function $f(x)$ is infinitely differentiable, then near a point a, it can be approximated by a polynomial whose coefficients are derived from the derivatives of $f$at a.

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{n}(a)}{n!}(x - a)^{n}$$

Using Taylor's theorem, we can approximate a function as a finite polynomial by taking a partial sum of its taylor series:

$$f(x) = \sum_{n=0}^{N} \frac{f^{n}(a)}{n!}(x - a)^{n} + R_{N}(x)$$

Deriving the Maclaurin Series for e^{x}:

The function e^{x} is particularly well-suited to this approach, as it has the remarkable property that all of its derivatives are identical:

$$\frac{d}{dy}e^{x} = e^{x},\ \frac{d^{2}}{dx^{2}}e^{x} = e^{x},\ \cdots$$

Evaluating these derivatives at $x = 0$ gives

$$f^{(n)}(0) = e^{0} = 1 \text{ for all } n$$

Substituting $f^{(n)}(0)$ into the Taylor series formula and setting $a = 1$, we obtain

$$e^{x} = \sum_{n=0}^{\infty} \frac{x^{n}}{n!}$$

or explicitly,

$$e^{x} = \sum_{n=0}^{\infty} \frac{x^{n}}{n!} = 1 + x + \frac{x^{2}}{2!} + \frac{x^{3}}{3!} + \frac{x^{4}}{4!} + \cdots$$

Convergence of the Taylor Series:

Consider the ratio of successive terms of this series:

$$\left|\frac{x^{n+1}/(n+1)!}{x^{n}/n!}\right| = \frac{|x|}{n} + 1$$

As $n \rightarrow \infty$, this expression tends to 0 for any fixed x. Therefore, the series converges for all real 0.

This is an extremely useful property, as $e^{x} = \sum_{n=0}^{\infty} \frac{x^{n}}{n!}$ for all $x \in R$

Number of Terms Needed to minimize error:

However, a calculator cannot sum infinitely many terms. Instead, it truncates the series after N terms, forming a Taylor polynomial. Taylor's theorem provides a bound on the error introduced by truncation. The remainder after N terms is given by

$$R_N(x) = \frac{f^{(N+1)}(\xi)}{(N+1)!}x^{N+1}$$

for some ξ between 0 and x.

For e^x, since all derivatives are e^x, the error is

$$\left|R_N(x)\right| \leq \frac{e^{|x|}}{(N+1)!}|x|^{N+1}$$

1. The factorial $(N + 1)!$ grows extremely rapidly, causing the error to shrink quickly as N increases.
2. The error also depends strongly on $|x|$: larger values of x require more terms.

Modern calculators typically use IEEE 754 double-precision (64-bit) floating-point numbers, which provide approximately 15–16 decimal digits of accuracy. For this level of accuracy, 15 terms are sufficient. However we need $|x|\leq 1$, as the series converges slowly.

Range reduction of x ≤ 1:

To overcome this, calculators use range reduction. The idea is to rewrite x in a form that brings it closer to zero, where the series converges quickly.

We use the identity:

$$x = nln\,2 + r$$

where n is an integer and r is chosen such that

$$r\leq \frac{ln\,2}{2}$$

Then,

$$e^x = e^{nln\,2 + r} = e^{nln\,2} * e^r = 2^n * e^r$$

Hence the value of r is reduced to between $0\leq 1$ and the majority of the computation is involving to 2^n, which can be computed efficiently in binary (via bit shifts).

3 Conclusion

The computation of e^x via the taylor series illustrates the power of approximation. Even a complicated non-polynomial function can be evaluated using techniques such as Taylor Series approximations and Chebyshev polynomials. Together, alongside other methods such as $ln\,x$ approximations via Chebyshev polynomials and $sin\,x$ calculations using the CORDIC algorithm, a calculator is able to evaluate such complex transcendental functions. I truly hope that overall, you have learned to trust your calculator and gained a deeper appreciation for the computation behind math, which is stranger than any other problem.