LETTERS FROM THE EDGE OF FOREVER

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Ramanujan's Mock Theta Functions

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Introduction: The Last Postmark

Imagine you are dying. You have weeks left, perhaps fewer. Tuberculosis has hollowed you out; the English winters have not been kind; you are far from home. Yet, you are not resting. You are writing mathematics.

This is Srinivasa Ramanujan in January 1920, thirty-two years old, in a rented room in Madras, India. He has already produced thousands of theorems during his short life – results so strange and beautiful that his mentor, G.H. Hardy, once said that Ramanujan's formulas must be true because '*If they were not true, no one would have had the imagination to invent them*'.

His final letter to Hardy contains something that would puzzle the greatest mathematicians on Earth for nearly a century. He lists seventeen mysterious functions. He calls them the *mock theta functions*. He says he cannot prove anything about them. But he is certain, with that absolute, peculiar Ramanujan certainty, that they are profoundly important and profoundly strange.

> "*I have constructed a new class of functions in the theory of numbers – the mock theta functions… they seem to have no modular equations.*"

Ramanujan died three months later, in April 1920, and the seventeen functions sat, unexplained for 80 years.

1. Waves, Patterns, and the Magic of Theta Functions

Before we can understand what is 'mock' about the mock theta functions, we need to understand what a real theta function is. And what better way to do this than to visit a pond.

Drop a pebble in calm water. You will see that a perfect ring of ripples spreads outwards. The height of the water at any point creates a wave, a sine or cosine function. Now, drop two pebbles simultaneously. The ripples from each source will interfere with each other, creating a lattice of peaks and troughs. This interference pattern is not random; it has a symmetrical mathematical structure.

Theta functions describe the structure of these repeating patterns. The simplest theta function, introduced by the German mathematician Carl G.J Jacobi in 1829, looks like this:

$$\theta_3(q) = \sum_{n=-\infty}^{\infty} q^{n^2}$$

This function, is known as the third Jacobi theta function.

Here, q is a number between 0 and 1, typically represented as:

$$|q| < 1$$

This function sums over every integer, positive, negative, and 0. The n^2 term means the powers are perfect squares.

Theta functions are everywhere. They describe heat flowing through a metal rod. They encode the vibration frequencies of drums and bells. In nature, whenever a pattern arises from the superposition of many periodic processes, the stripes on a zebra, the arrangement of seeds in a sunflower, the crystalline structure of a snowflake, theta functions are present, creating order.

What makes theta functions genuinely remarkable, however, is not just that they appear here. Theta functions are characterised by a property called modularity. Think back to the lattice of ripples in the pond. If you shifted your perspective by one full ripple, the pattern would remain identical. This is known as periodicity. Modularity, is a much deeper symmetry, which periodicity is often a precondition for.

To better explain modularity, imagine the pond's surface isn't just a flat sheet of water, but a flexible fabric. Modularity means that the interference patterns stay the same, even if you grab the centre of the fabric, and pull it towards you, turning it 'inside-out'.

We can represent this transformation as follows:

$$\theta(q) \approx \theta(1/q)$$

We can therefore understand modularity to be 'when we pull the fabric inside-out, and the patterns stay the same'.

From this, you can understand mock-theta functions as being theta functions, but with a small tear in the fabric, which when flipped inside out, causes the function to have a slightly different pattern. This can be fixed by adding a small patch of material to the tear, and when fixed, the mock theta function would become a theta function.

2. What Is a Mock Theta Function?

What better way to learn more about mock theta functions, than by studying one.

$$f(q) = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2 \cdots (1+q^n)^2}$$

This is Ramanujan's first mock theta function, often called $f(q)$.

Don't let it scare you! Compare it to the Jacobi theta function. Both are very similar, involving powers of q. If you only glanced at them, you might think they were cousins. However, the squared denominators in $f(q)$ change everything.

Let us return to the pond, now a fabric. A true theta function, when you pull it inside out, produces a pattern identical to what you started with. Mathematicians say it *transforms nicely under the modular* group. This is a very useful property, as it means you can use one theta function to understand infinitely many others, as they are all related by those symmetries. Ramanujan found that $f(q)$ does not do this. Rather annoyingly, there is a small shift in the pattern, caused by that tear in the fabric, that means it does not transform this way. Ramanujan could not pin down what was missing, or how to fix that tear in the fabric. He wrote to Hardy, '*Unlike the 'real' theta functions, they seem to defy the modular equations*'.

This is the main mystery at the heart of the mock theta functions. They look like they should obey the rules, but they don't. Or rather, and this took 80 years to understand, they obey a different set of rules. They are therefore not fake theta functions, they are a new kind of mathematical object entirely.

3. Eighty Years of Mystery

After Ramanujan's death, some of the best minds in number theory tried to crack the puzzle of the mock theta functions. Hardy worked on them. G.N. Watson, in his 1936 presidential address to the London Mathematical Society, called the mock theta functions '*a gift from heaven*' and devoted much of his career to proving their results. But still, no one could explain what they were.

This difficulty stems from how every ordinary theta function belongs to the theory of modular forms. A modular form is a function that satisfies the inside-out symmetry that we defined earlier. This underlies Andrew Wiles's 1995 proof of Fermat's Last Theorem, one of the greatest achievements in the history of mathematics.

Mock theta functions refused to fit into this framework. They were too irregular to be modular forms. Yet they were clearly not random – their coefficients had patterns, and their properties were eerily consistent. They seemed to belong to some theory that did not yet exist.

Generations of mathematicians struggled with them, proving some facts, and cataloguing their peculiarities. They found deep connections to partition theory (counting the number of ways to break a number into smaller pieces), to combinatorics, and to the Rogers-Ramanujan identities. But no-one could understand the bigger picture.

> *"[Ramanujan] discovered so much, and yet left so much more in his garden for other people to discover."*

4. The Shadow: Zwegers and the Resolution

In 2002, a Dutch PhD student named Sander Zwegers had a remarkable insight. He didn't ask about what was wrong with the mock theta functions, what tears they had, but rather he was the first one to ask, what material can I add to the tear to fix it, what are they missing?

His answer: a shadow.

Let us return to the pond-turned-fabric one more time. Imagine that our pond's surface is not just 2 dimensional, but casts a faint shadow onto the ground beneath it. The fabric, with a tear, when flipped inside out, changes slightly, but, if you consider both the fabric and its shadow together, imagining the whole solid to consist of both the fabric, and the shadow, the whole solid remains the same, and transforms perfectly. The shadow is precisely the patch of material needed to repair the tear.

This is essentially what Zwegers showed. Each of Ramanujan's mock theta functions is missing a correction, the shadow. Alone, it cannot satisfy the conditions for modularity, but paired with its shadow, the resulting whole solid transforms perfectly. In mathematical terms, Zwegers proved that each mock theta function is the holomorphic part of a harmonic Maass form, the smooth fabric, part of the entire solid. This is known as a mock modular form, and is a generalisation of a modular form that satisfies the Laplace equation:

$$\Delta f = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f = 0$$

The Laplace equation governs heat diffusion, electrostatics, and fluid dynamics – the same patterns that theta functions were already known to describe. Zweger's result, developed further by Ken Ono, Jan Bruinier, and others showed that Ramanujan's functions were not anomalies, rather they were the first glimpses of a new landscape of objects, known as mock modular forms, the fabric, that had been hiding in plain sight.

Ramanujan, whilst on his death bed, had intuitively understood the layout of this landscape, 82 years before its map was drawn.

5. Why Does Any of This Matter?

It is very tempting to believe that mock theta functions are simply mathematical curiosities – beautiful, abstract, and remote from the real world. But this could not be further from the truth. Remember the Laplace equation, that equation which harmonic Maass forms satisfy. That equation which governs heat diffusion, electrostatics and fluid dynamics.

The shadow that repairs that tear in our fabric is not some invented mathematical patch. It is a structure that nature has been using for eternity. This connection runs extremely deep. In string theory, physicists need to count the number of quantum states of a black hole of a given mass. These counts can be found within the coefficients of mock theta functions, as the shadow term corresponds with contributions from holes in spacetime.

A dying mathematician in Madras, one who has never heard of String theory, or quantum mechanics or black holes has written down the exact functions used to count states at the edge of the universe. This is not a coincidence. Ramanujan was not just a mathematician. He mapped a new landscape, he is a geographer. The mock theta functions showed that this landscape exists.

And the most mysterious question is not mathematical at all, but rather, how did he know?

Conclusion: The Unfinished Letter

Mathematics is often described as a finished art. You read a problem, you write down a definitive proof. It is certain and complete. The story of mock theta functions suggest otherwise. Ramanujan wrote down seventeen functions, and confessed he could not prove why they behaved as they did. The greatest mathematicians of the 20th century could not explain them. And yet the functions were not wrong. They were right, the patterns were real. The mystery was not that Ramanujan was wrong; rather it was that the rest of mathematics had not yet caught up with him.

My personal fascination with the mock theta functions stems from how they are 'almost' right, they are not certain. Mathematics is usually presented as certain – you prove a theorem, and it is true, and remains true forever. There is something deeply moving, somewhat poetic about the fact that Ramanujan's last letter was uncertain. He trusted his intuition over his proofs. And his intuition, after a lifetime of work in number theory, was right.

The seventeen mock-theta functions he listed are now fully understood, and many more have been added to this list. Not bad for a letter never meant to be a final word.

Further Reading

I strongly recommended the following for anyone who wants to explore the topic further.

Hardy, G. H. (ed.). *Collected Papers of Srinivasa Ramanujan*. Cambridge University Press, 1927

Zwegers, S. *Mock Theta Functions*. Universiteit Utrecht, 2002.

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Singh, S. *Fermat's Last Theorem*. Fourth Estate, 1997.

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