The “Art” of Abstraction

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1 The Problem

18th October, 2025. The day before the Louvre Heist. You’ve just been appointed as the new Chief Security Officer of the museum, and you are tasked with ensuring the protection of the adored French Crown Jewels exhibit by stationing guards around the room. However, budget constraints are tight, and your goal is to achieve total coverage using the minimum number of guards possible. How would you accomplish this?

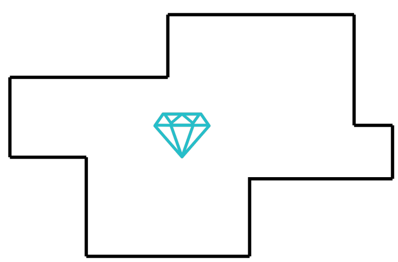

Figure 1: The Art Gallery

2 Tackling the Problem: Basework

Daunting, right? Surely there’s too much to consider. What if the layout is irregular? What if lighting is poor? What if one guard has worse vision than another? Strip that all away.

Throughout this essay, I want to demonstrate the power of simplification. In many problems, mathematical or not, by first answering the question “What actually matters?”, we can gain a clear direction in which we can traverse through our problem.

So, what is essential here? We answer this by fixing a set of assumptions that remove unnecessary complexity.

- Firstly, we must define visibility. We can define a point as "visible" to a guard if the line segment drawn from the guard to a point completely lies inside the museum.
- Secondly, let us define how guards will behave. Let us impose the condition that guards must be stationary, can stand anywhere, and each has 360 vision (similar to a lightbulb).
- And finally, we can assume the museum to be "guarded", if all points of the museum are visible to at least 1 guard.

With a clear set of assumptions, our problem begins to look far more approachable.

3 Beginning to Form a Solution

However, even with a more *approachable* problem, our next step may still look unclear. So again, we think about simplifying the problem. In this case, we can consider thinking about the most simple case; convex polygons.

A convex polygon is defined as a "polygon in which, for each pair of points A and B, the line segment connecting A and B completely lies within the polygon" (i.e each pair of points are visible to each other), or, less formally, we can consider it to be one where each interior angle is less than 180°.[1]

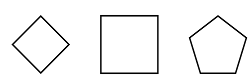

Figure 2: Convex Polygons

Truly, reader, I implore you to pick any two points in these given polygons, and try and see if you can break this rule.

As we know every pair of points are visible in this polygon, we know if we place a guard in any point of a convex polygon, by definition, this guard will be able to "view" each other point in the polygon, and therefore guard the museum.

Therefore, we have arrived at our first *lemma*[2]

[1] Note, the former definition is a more formal requirement that *leads* to each angle being less than 180°

[2] A fancy way of saying an intermediate theorem which will be used for our proof

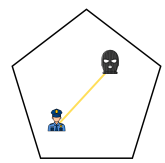

Figure 3: Guards!

- ***Lemma 1:*** *In any convex polygon, one guard may suffice to guard it.*

Great! We now have an avenue which we can explore. If we can simplify our problem of "guarding one complex polygo**n**" into one of "guarding multiple convex polygon**s**", we will be well on our way to solving our problem.

4 Exploring this Route

Exploring this route further, we must first ask the question: is this avenue we have discovered possible? Is there a simple way to reduce this problem of guarding a complex shape into one of guarding multiple convex polygons?

Often in solving a problem, we must tread with caution; discovering a new avenue mustn't always imply that the route will lead to the results we desire.

With this in mind, what convex polygon could we try decomposing our shape into? Let us *again* think about the simplest case (you may be spotting a pattern here); what about the humble triangle? Being the polygon with the least number of vertices, perhaps it will yield the simplest solution? And, as the sum of angles in a triangle must always add up to 180°, each angle must strictly be below 180°, therefore we know all triangles must be convex!

With a bit of experimentation, it does *seem* possible. Again, consider these examples; can you identify a different way to decompose each shape into triangles?

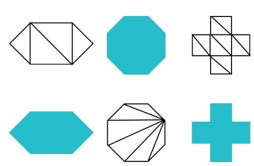

Figure 4: Triangulations

But now, how can we *prove* it is possible?

4.1 Induction

Moving from intuition to proof is often a challenging task. Even statements that feel extremely intuitive, such as the 4 colour theorem, can have really complex proofs. To prove this statement, I want to introduce a powerful technique; **induction**.

Induction relies on a similar reasoning pattern to ours, one that emphasises first considering the simplest case. To use it, we prove a statement for a "base case"[3], then prove that if a statement holds for one case, it must hold for the next case.

To show that this is enough to prove ***all*** cases, say, I tell you that: "If Joseph eats spaghetti on one day, he will, without a doubt, eat spaghetti on the next". Now, if we know Joseph eats spaghetti on Monday, can we prove that Joseph will eat spaghetti on Sunday?[4]

Yes! If Joseph eats spaghetti on Monday, he will eat spaghetti on Tuesday. And if he eats spaghetti on Tuesday, he will eat spaghetti on Wednesday; repeating this chain of reasoning until we reach Sunday. Therefore, this information does suffice!

We will use a variant of this "domino-style" reasoning called "strong induction" to prove that all shapes can be decomposed into triangles. Here, instead of using the fact that if just the current case holds true (i.e Joseph will only eat spaghetti on the next day if he ate in on the current day), the next holds, we use the fact that if all previous cases hold true (i.e Joseph will only eat spaghetti on the next day if he ate it on all previous days of the month), the next case holds.

- ***Claim**: Polygons of any number of vertices can be decomposed into triangles.*

Let's build up our inductive proof starting with the simplest case: the triangle. Naturally, a triangle can be decomposed into triangles, therefore the claim holds true in the base case.

Now, let us first assume this statement holds true for all previous cases up to and including the polygon with n vertices (i.e, if n = 5, we are assuming a polygon with 4 vertices can be triangulated and a polygon with 5 vertices can be triangulated).

If it holds for previous cases, if we can split our shape into two shapes with less vertices, we can then triangulate each separately and combine.

We do this by constructing a diagonal in our gallery. We first find an angle less than 180° (which must exist, as the sum of angles is $(n-2)*180$, therefore if all

[3] Which is often the most simple case

[4] Well, assuming a few things like ""Joseph will not perish within this time period", or, "No one, not even Joseph's concerned doctor, can prevent him from eating his daily dose of spaghetti"

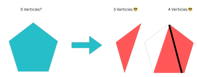

Figure 5: Decomposing a shape

n angles were 180 or more, this would not be possible as $n * 180 > (n-2) * 180$). We then draw a diagonal between this angle's two neighbouring vertices.

If this diagonal completely lies inside the shape, we can split the shape along this diagonal to attain two shapes we can triangulate as desired.

More interestingly, if not, this implies at least 1 vertex lies inside the triangle formed between the diagonal, its endpoints, and the vertex of the angle. From here, we can move the line segment perpendicularly until it hits the last vertex in the triangle, and draw a line between this vertex and the vertex of the angle, which must now lie inside the polygon. We now have a new diagonal we can split the shape along, and attain two shapes to triangulate.

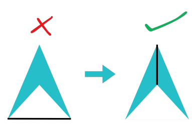

Figure 6: Trying a new diagonal

As we have proved the base case, and we have proved that if this claim holds for all previous cases, it holds for the next, we can use our domino-style reasoning to prove our claim.

With this, we arrive at our second lemma;

- ***Lemma 2:*** *Any polygon can be triangulated.*

5 Placing our Guards

This is looking very promising. We now have a figure which we know we always can reliably protect.

Now, you might be tempted to say: "the number of triangles we form will be our minimum as each triangle is convex, therefore if we place a guard in each triangle, our museum is now guarded".

However, again, remember we are trying to optimise! Ideally, we want to use the

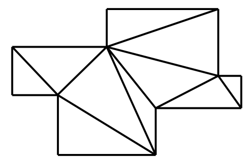

Figure 7: Our new gallery

least guards possible, therefore having each guard "protect" just one triangle might not be efficient.

Let us consider each position where we can place each guard in each triangle. Notice what happens when we place the guard completely inside the triangle. In this case, we can only ever "guarantee" that the guard protects the triangle it sits in - perhaps not the best use of our personnel.

However, what if we instead place a guard on the edge between two triangles? Depending on our choice of edge, we can now "guarantee" that a guard entirely protects two triangles at once! And better yet, what if we consider a vertex? In the best case scenario, a guard might be able to protect an infinite number of triangles!

As placing a guard on a vertex leads to the highest "potential protection", and in all cases where placing a guard on an edge would protect two triangles, placing a guard on a vertex would achieve the same, we can thus conclude that our solution will involve some sort of arrangement of guards on vertices.

So, again, our problem is simplified to the question: **what is the best way to place guards on vertices**?

5.1 Graph Colouring

And again, this question is complex, so let's just consider one triangle of our choice. Ideally, you'd only want to place a guard on only one vertex, as after you've placed one guard, the triangle is protected.

There are three possible places to place the first guard, hence, to visualise this[5], let us colour the three possible initial positions with red, blue, and green respectively.

With this in mind, what is the simplest next step?

Ideally, it would be deciding guard placement in the next adjacent triangle, as we have clear restrictions! We wouldn't want to place our next guard on vertices

[5]Note, the way we label our possible "guard sets" is arbitrary: we could've easily chosen A, B and C or apples, bananas, and strawberries. Colouring simply leads to a nice way to visualise the situation after we finish our construction.

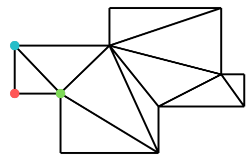

Figure 8: Our new gallery, coloured

which also lie on our first triangle, as then our first triangle will be protected by 2 guards, which may be redundant.

Moreover, by traversing through the graph like this, we won't run into any contradictions as we won't visit each individual triangle more than once (since the structure branches like a tree, so once we move forward we never return to an earlier region).

Let's think through our placement at the next triangle. If we started with the red vertex, our next guard would be placed at the opposing corner of the next triangle, as this is the one in which we guarantee no redundancy. If we started with the blue vertex, we wouldn't place any new guard, as our new triangle is already protected. This is similar for the green vertex.

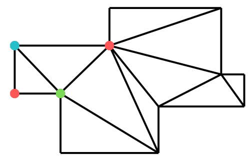

Figure 9: Our new gallery, coloured

We can repeat this process for each triangle of the graph. Now, what do we notice about our result? In each triangle, each colour, representing a different pattern of placing guards, appears exactly once.

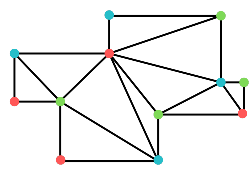

Figure 10: Our new gallery, coloured

Therefore, if we consider just one set of colours (corresponding to the set of guards which starts from a set starting position), each triangle has one vertex with a guard, therefore each will be guarded, and therefore our museum is guarded.

Once we've obtained this, we can simply pick the starting position that yields the lowest number of guards.

6 Finalising: The Best Upper Bound

So, how many guards does this translate to? Altogether, we know the sum of the red, green and blue coloured vertices must be n, the number of vertices of our shape (r + g + b = n).

Assuming our guards are distributed uniformly, each would have a value of $n/3$. Through experimentation, we can conjecture that in all cases, at least **one** group has $\leq n/3$ vertices.

To prove this, we can use contradiction:

Assume:

$r, g, b > n/3$

Therefore:

$r + g + b > n$

But:

$r + g + b = n$

Therefore our assumption must be false, so one of r, g, or b is $< n/3$. However, as each must be an integer, we can add the stricter condition that either one is $\leq \lfloor n/3 \rfloor$!

Hence, we have a bound of guards which we know we can always apply to **any** complex shape. And to assure that this is the best bound (i.e our number cannot get lower to work in all cases), consider this shape: as this is a shape which requires $\lfloor n/3 \rfloor$ guards, this bound is the best we can get.

We have now arrived at our result:

- ***Art Gallery Theorem:*** *For any art gallery, $\lfloor n/3 \rfloor$ guards are sometimes necessary, and always sufficient, to guard the gallery.*

7 Conclusion

What a journey! With many problems in discrete maths, I believe it's far too easy to look at how "simple" our result looks, and underestimate the process to reach the answer. It's true, we didn't utilise "Atiyah-Singer's Index Theorem",

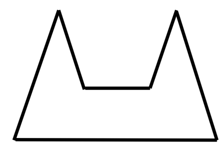

Figure 11: Extreme Case

or the "Gross-Pitaevskii Equation"; all it took were arguments built on a simple understanding of geometry.

But truly, look how rich our reasoning was! Although our path was slightly lucky - proving a theorem often takes far more "wrong" avenues before we reach the answer: in fact, if we consider this problem in a 3D space, it still remains unsolved - look how many arguments we needed to make to come to the elegant result of $\lfloor n/3 \rfloor$!

And in fact, we have utilised complex concepts in our *reasoning*: we utilised **abstraction** - eliminating unnecessary details from our problem - and **decrease-and-conquer** - reducing our problem to its simplest case, solving it, then extending our solution to the original problem.

This is why I find discrete maths so beautiful, although results often seem "simple", the reasoning to attain them can be deceptively challenging, intricate, and enamouring.

If anything, as we conclude this essay, I hope you'll leave with everything **but** the result. I hope you'll leave with a new mentality, one which will allow you to decompose complex problems and approach them with confidence. I hope you can apply these concepts to problems you face, whether in mathematics class, or when facing the many problems life proposes. And, most importantly, I hope you can look at this solution, and not see it as a collection of arbitrary arguments, but rather ones you could have derived yourself through a strong line of reasoning.