M ∩ A, where *M* is mathematics and *A* is art.

1. Introduction

Where lies a boundary between exact and imaginary? Throughout the history the world has seen many intersections between the fields of art and mathematics: use of golden ratio by Rebrandt, depiction of polyhedras on mosaic floor of 15th century in St. Mark's Basilica in Venice or the whole art movement of cubism that focused on showing reality in geometrically simplified shapes. As someone interested in art as well as in mathematics, I really wanted to learn more ways in which those two subjects can be combined.

Fig.1. Rembrandt "The Anatomy Lesson of Dr. Nicolaes Tulp".

Fig. 2. Picasso "Girl with a Mandolin (Fanny Tellier)

While looking at different examples of connection between those two fields, my interest was captured by Maurits Cornelis Escher – a 20th century artist, who got famous with graphical works with impossible figures, patterns and geometrical figures. His works fascinated me, so I was eager to explore the mathematical concepts behind his artworks. This essay is an attempt to explain myself the properties of those shapes behind Escher drawings and how to derive them myself.

2. Tesselations

One of the first things that a viewer will notice going through a gallery are Escher's pattern designs that are made with mathematical precision. Examples of such works are "Development I", "Regular Division of the plane with birds" and "Cycle" (Fig. 3, 4, 5).

Despite all shapes look very complicated, they are made based on simple regular geometric shapes such as squares, triangles and hexagons. If we try to tesselate the plane ourselves, soon we would find that those shapes are the only suitable ones. But why? Let's try to prove it mathematically.

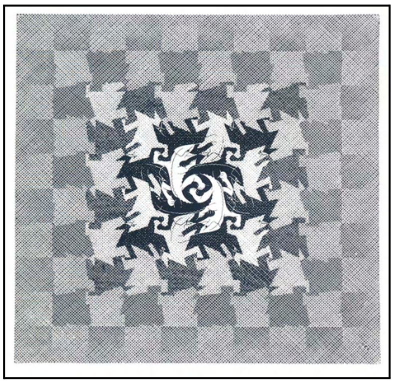


Fig.3. Escher, “Development I”.

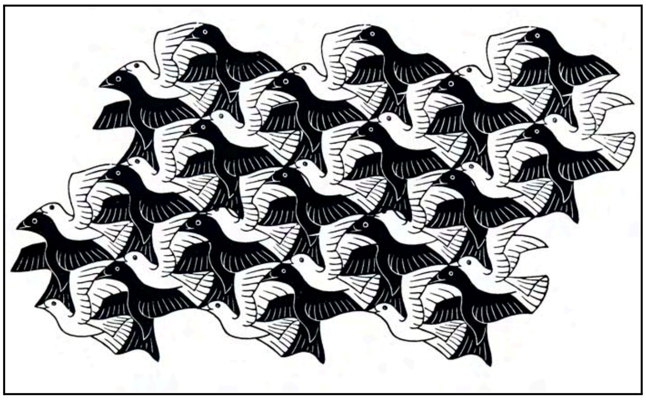


Fig.4. Escher, “Regular Division of the Plane with Birds.

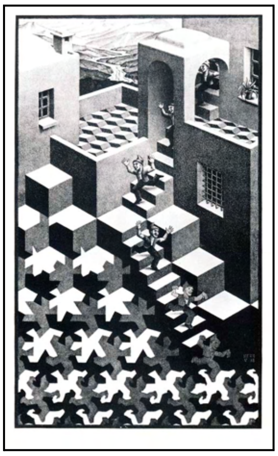
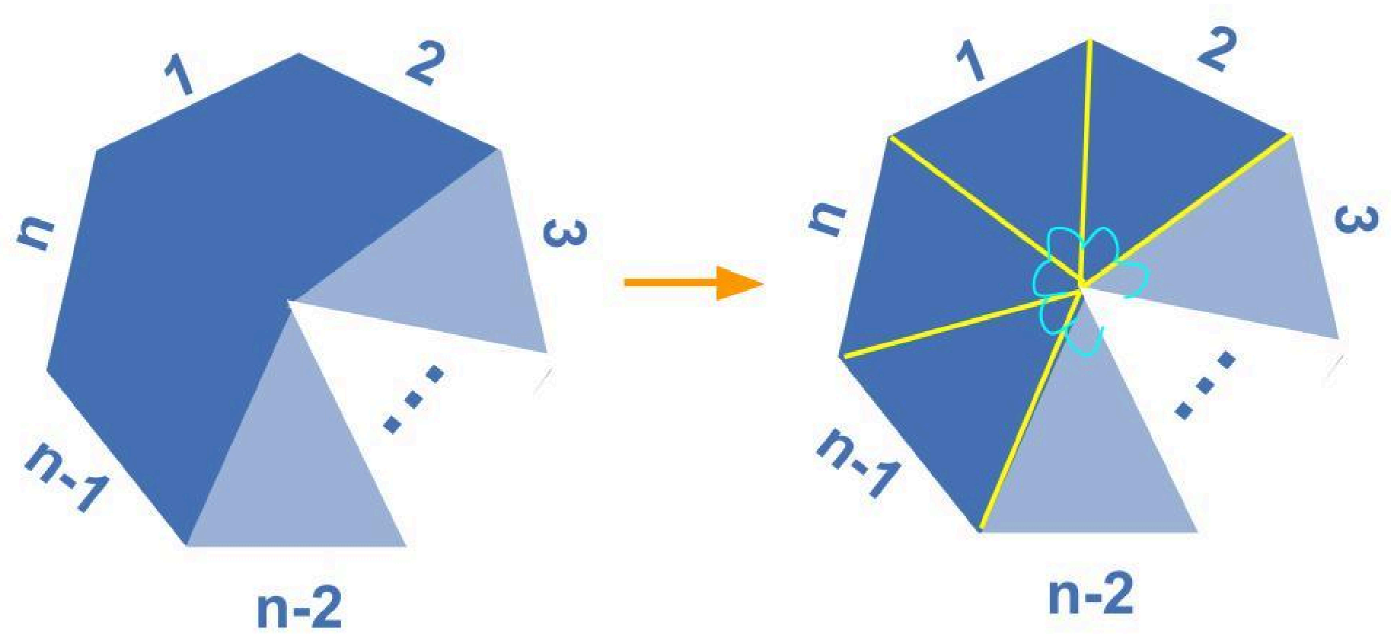


Fig.5. Escher, “Cycle”

Let’s start with finding the total sum of inside angles in the regular n-sided polygon and firstly divide our shape on triangles with bases on each side.



This way we get n triangles with the sum of angles 180. Now we can just eliminate those angles in the middle of the polygon (with a total sum of 360°):

$$\text{total} = n \cdot 180 - 360$$

$$\text{total} = n \cdot 180 - 2 \cdot 180$$

$$\text{total} = (n-2) \cdot 180$$

Hence, one angle in is equal to: $\frac{(n-2)\cdot 180}{n}$

If we go back and look at tessellations, where m is the amount of tiles touching each other in one vertex, we get that:

$$m \cdot \frac{(n-2)\cdot 180}{n} = 360$$

$$m \cdot \frac{180n - 360}{n} = 360$$

$$180m - \frac{360m}{n} = 360$$

$$\frac{m}{2} - \frac{m}{n} = 1$$

$$m\left(\frac{1}{2} - \frac{1}{n}\right) = 1$$

$$\frac{1}{2} = \frac{1}{m} + \frac{1}{n}$$

$$m \cdot n = 2(m + n)$$

$$m \cdot n - 2(m + n) = 0$$

$$m \cdot n - 2m - 2n = 0$$

Add 4 to each of the sides:

$$m \cdot n - 2m - 2n + 4 = 4$$

$$(m - 2)(n - 2) = 4$$

$m, n \in \mathbb{Z}^+$ and the only multipliers of 4 are: $4 = (4 \cdot 1), (2 \cdot 2), (1 \cdot 4)$. As a consequence the only fitting pairs of m and n are:

	m	n
(6 – 2)(3 – 2) = 4 × 1 = 4	6	3
(4 – 2)(4 – 2) = 2 × 2 = 4	4	4
(3 – 2)(6 – 2) = 4	3	6

Hence the only regular figures fitting for tiling are triangles, squares and hexagons. These tilings can be visible on the works of Escher: in “Development I” square tessellations are used, where the artist showed himself how lizards are made out of squares, while “Regular Division of the plane with birds” uses triangular tessellations:

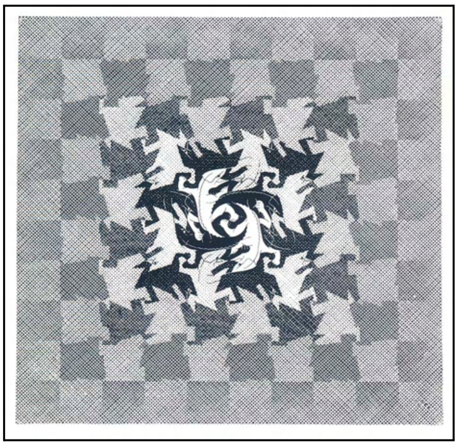


Fig. 3

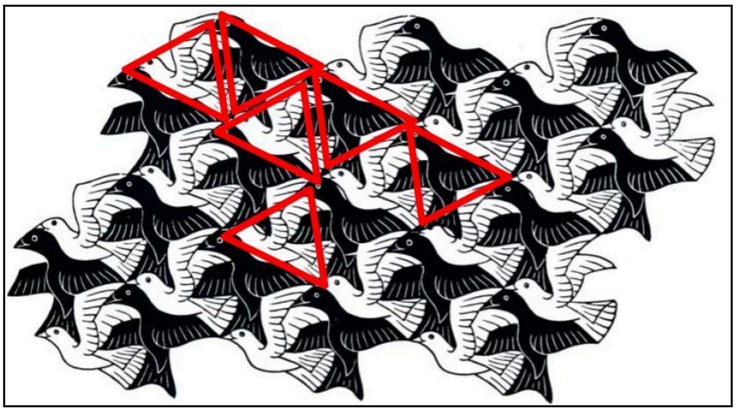
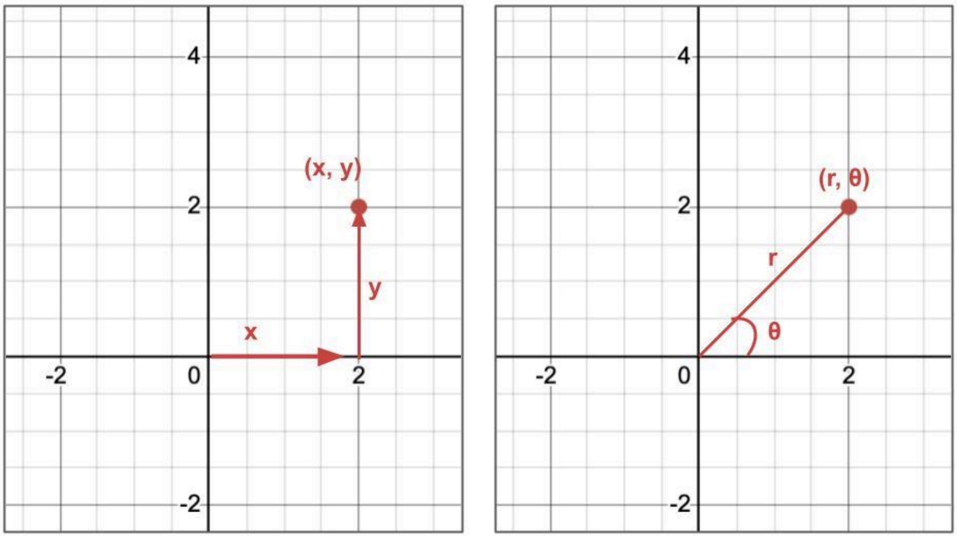


Fig.4

3. Logarithmic spiral

What is it and where can we find it in artworks of Esher? Let's start from afar with the Cartesian plane. Normally we use coordinates among x and y values, making x = f(x) and the point being described by (x, y). However another method of noting coordinates is through angle (θ) from a chosen point on a plane, usually its origin, and the radius (r). This way the points are written as (r, θ) – polar coordinates, and r = f(θ).



It is important to remember that those two systems of coordinates don't have the same coordinates for the same point. As an example from above:

$$(x,y) = (2,2)$$

However, for (r, θ) for θ in degrees:

$$\left(\sqrt{2^2+2^2}\,,\ \frac{180-90}{2}\right)$$

$$\left(2\sqrt{2}\,, 45\right)$$

The equation for logarithmic spiral however is ln(r) = aθ, which is usually expressed as:

$$r = e^{a\theta}$$

Where a is constant and determines the spiral's rate of growth. Fascinating, but with a change of θ, r is growing exponentially, because $e^{a(\theta+b)} = e^{a\cdot\theta + a\cdot b} = e^{a\theta}\cdot e^{ab}$, where b is a change from previous value, hence r will be increasing by factor e^{ab}.

The logarithmic spiral was used in art by many artists including Escher:

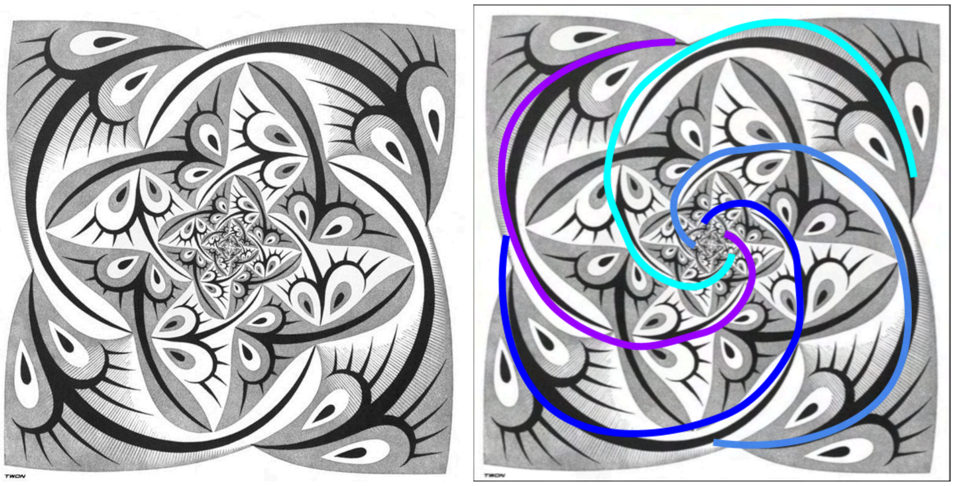


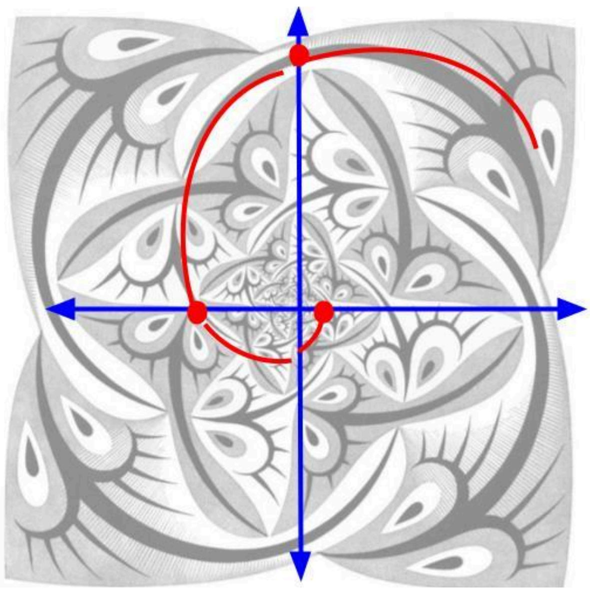
Fig. 6. Escher, "Path of life II)

But the question stays unsolved: what is the function for r? In our equation the only unknown is *a*. We can notice that as angle increases, the function decreases, which means that a < 0, hence:

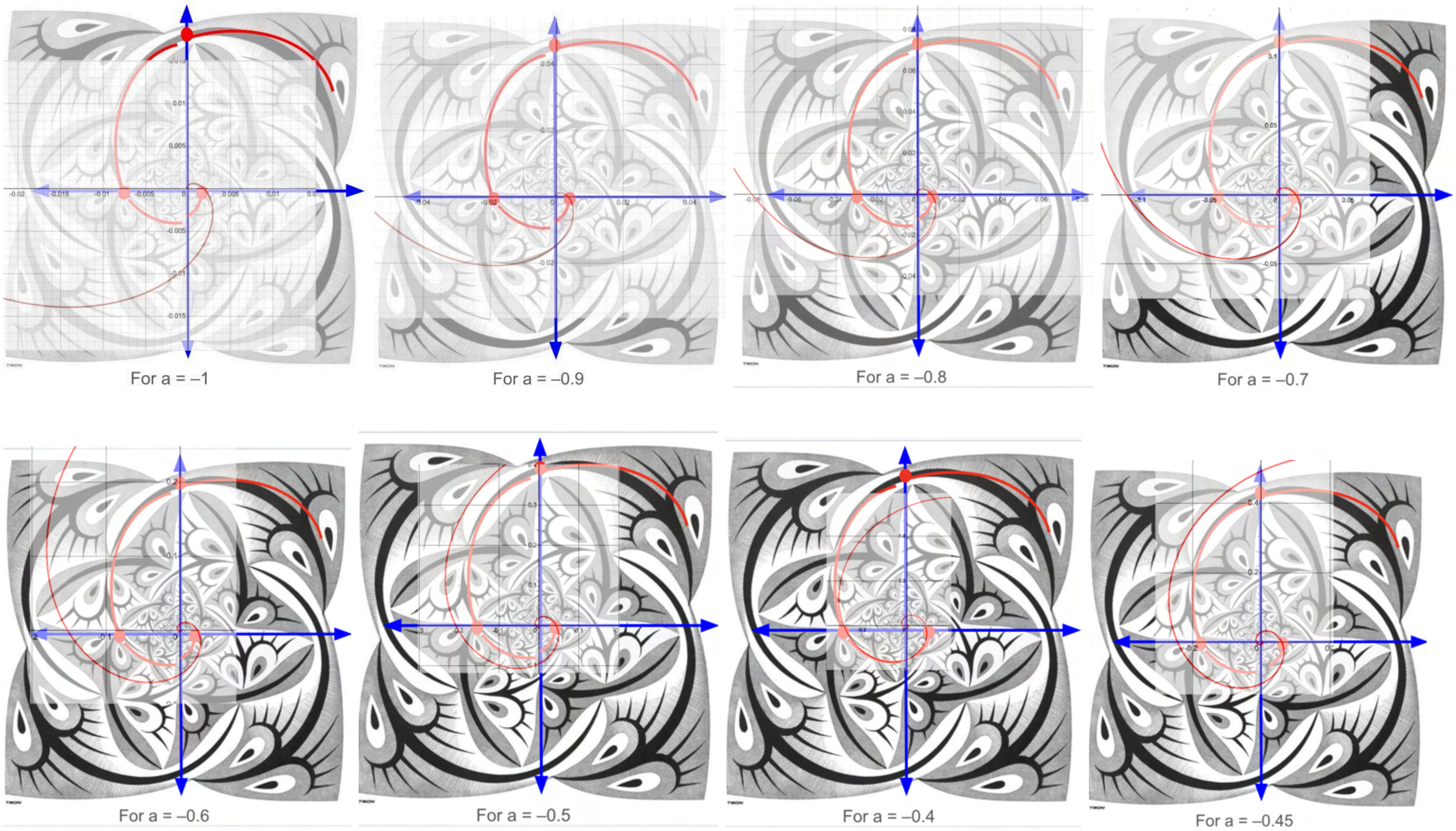
$$r = e^{-a\theta}$$
$$r = \frac{1}{e^{a\theta}}$$

The function will be decreasing all the time, and it's biggest value will be at θ = 0, r = 1:
$r = e^{a\theta} = e^{0a} = e^{0} = 1$

because after it θ > 0, hence value a×θ will be increasing, hence the fraction $r = \frac{1}{e^{a\theta}}$ will be smaller and smaller as θ increases:



Well, let's try to match this diagram to possible solutions.

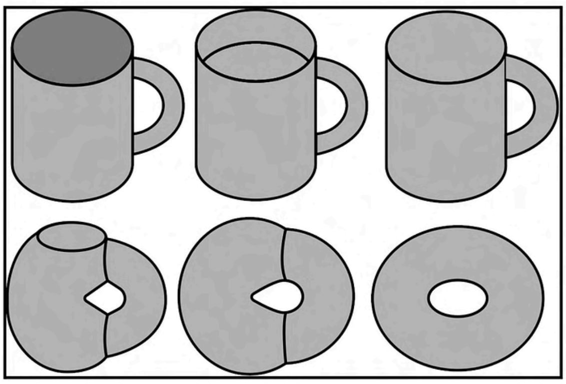


Through trial and error it has been found that one of possible solutions lies in the range of —0.5 < a < —0.4.

4. Topology

Topology is a relatively new branch of mathematics that appeared in 20th century. It explores properties of spaces that cannot be changed under any continuous transformations (University of Waterloo). The examples of such transformations can be stretching, bending. However tearing, gluing new parts or creating new holes are not considered continuous transformations.

It is possible to say that square is topologically equivalent to the circle, and a cup is topologically equivalent to torus.



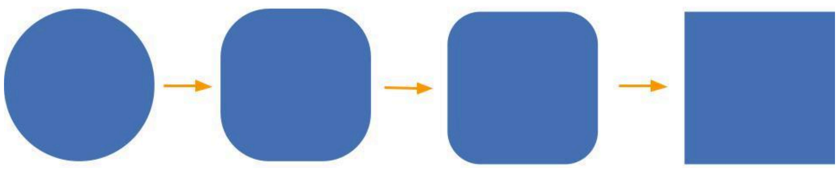


Fig.7. Formation of a Donut from a Coffee Mug

If through such transformations on the object you get another one, those two spaces are considered to be homeomorphic. Algebraically this means that there is such function *f* from the sets of points on the first object to the second one, that it is one-to-one and onto.

4.1. Homology

What exactly are those properties that can't be changed by continuous transformation? One of them is the amount of holes that an object has, which can be explored through homology. According to Eric Weisstein "a hole in a mathematical object is a topological structure which prevents the object from being continuously shrunk to a point". Through homology we can determine how many holes we have in n-dimensional space. Here you won't find a full exact mathematical definition of what is homology, but rather its simplified version that was easier to digest for myself and explain to others. Well, let's look at what exactly is homology on an example of one of the shapes depicted by Escher — Möbius strip:

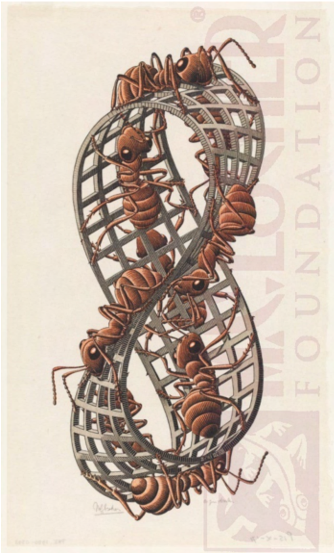
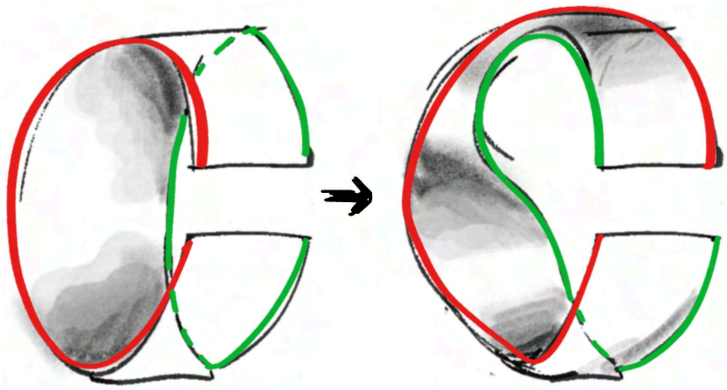


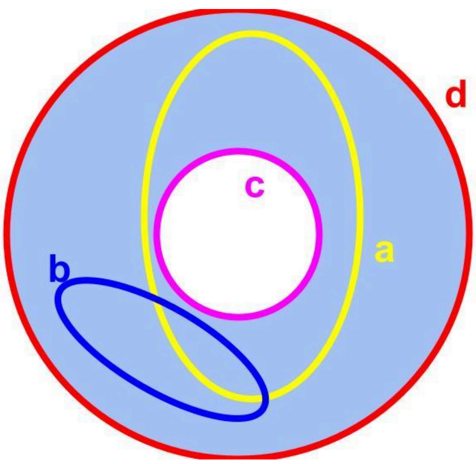
Fig.8. Escher, Maurits. Mobius Strip II.

You can make one yourself by just flipping one of the sides and gluing it to an opposite side:



The object that we get, unlike a rectangle it was made of, has only one side. This was wonderfully illustrated on Fig. 8 — if you follow the path that ants take, your eyes will return to the places they started.

But how are we going to find holes, if we don't even have an exact definition of what a hole is? Well firstly, it must be a circle, otherwise what kind of hole even is this? But what is the difference between just a loop on an object and a hole?

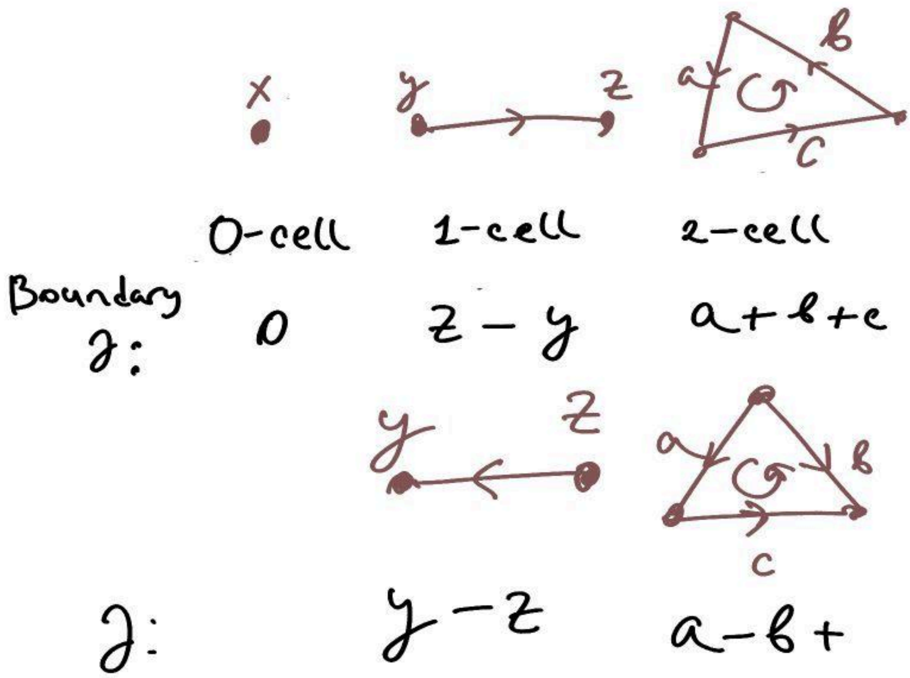


Unlike a or b, c and d contain a fully separated space inside of them. However the difference between those 2 loops is dramatical – d is a boundary of a 2D space, while c – of empty one.

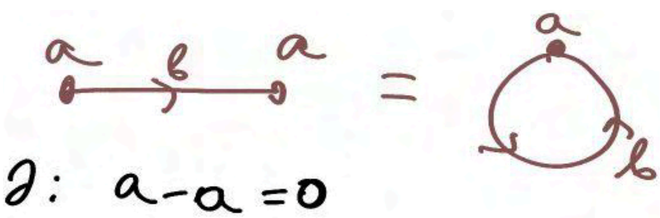
Alright, but, as true mathematicians, we are going to complicate our life even more and ask quite a philosophical question ~~— to be or not to be~~ what is a loop?

Let's start with breaking the spaces into smaller parts called cells. The simplest shape is in 0 -dimensional space — a dot called 0-cell. The next level would be a line — 1-cell, the next one is 2-cell as triangle, and etc. But in order to find holes, we need to find boundaries, notated as δ_n , on those cells.

To clarify for a 2-cell: in order to find direction we need to set a rotational arrow. Sides that go in the same direction have a " + " sign, but the ones that go in the opposite – have a "—"sign.



But what would a loop look like in terms of cells?

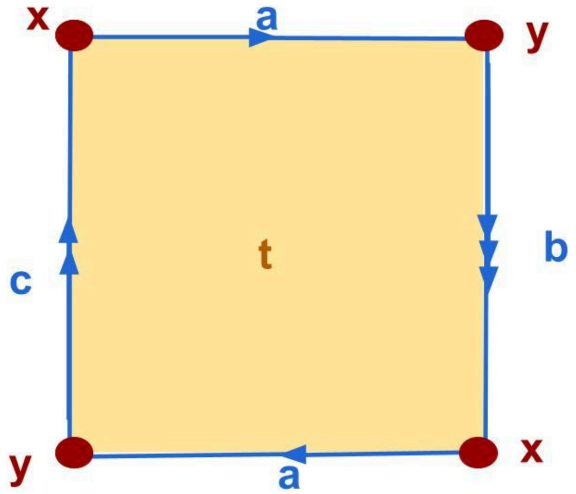


From this we can finally define what a hole is — a 1-cell with boundary 0, that is not a boundary of any 2-cells. HUZZA, one boundary behind on a way to what is homology (Funny right? Cause boundary. Sorry. Let's get back to serious mode).

Every object is built out of those figures. Meaning that there is a continuous function from elements of (n-1)-cell into n-cell. Hence, when we are trying to find how many holes each space contain, we can start backwards using reverse function (called image): from n-dimensional space into 0-dimensional space through such functions as

$$0 \rightarrow C_n(X) \rightarrow C_{n-1}(X) \rightarrow \dots \rightarrow C_1 \rightarrow C_0 \rightarrow 0$$

Now let's try to apply our knowledge of boundaries and holes on the Möbius strip. Since it is just a rectangle, which was twisted and glued together, let's put it back to its original form for simplification.



We will use $\partial_n : C_n(X) \to C_{n-1}(X)$ to show that ∂_n is an image of a function $C_{n-1}(X)$ into a higher dimensional cell.

$$0 \to C_2 \to C_1 \to C_0 \to 0$$

faces edges vertices
2-cell 1-cell 0-cell

$\partial_3 : C_3 \to C_2$, $C_3 = 0$
$C_2 = t$
$0 \to t$

Here, because C_3 is 0, ∂_3 is also equal 0, meaning that no holes on 3-dimensional space.

$$\partial_2 : C_2 \to C_1$$
$$t \to a + b + a + c$$
$$\partial_2 : t \to 2a + b + c$$

Interestingly enough, through a chain of complex operations we might get that the first homology group of this Möbius strip of X is equal to 0. “Might”, because due to the word limit the continuation of this wonderful hole-investigation won’t be here.

5. Conclusion:

With this essay coming to an end, there is so much that I would love to tell you more about, but the word count is fierce and relentless and so I’m afraid we must stop at this point. However, I hope one day I will be able to present to you more mathematical secrets hidden behind silent artworks, waiting to be found. I would love to dig deeper into topology, other properties of impossible shapes, Poincaré’s disk model, which was used so often in the works of Escher and many other fascinating concepts.

At the end of writing this essay, I hope you, my beloved reader, found new mathematical expressions or an original way of its application in art just as I have, and shared this joy of discovering new properties of patterns and objects with me.

Figures:

Figure 1: van Rijn, Rembrandt. *The Anatomy Lesson of Dr. Nicolaes Tulp*. 1632. Oil on canvas. Mauritshuis museum, The Hague.

Figure 2: Picasso, Pablo. *Girl with a Mandolin (Fanny Tellier)*. 1910. Oil on canvas. MoMA, Manhattan.

Figure 3: Escher, Maurits. *Development I*. 1937. Woodcut. The Palace Museum, The Hague.

Figure 4: Escher, Maurits.. *Regular Division of the Plane with Birds*. 1949. Wood engraving. The Palace Museum, The Hague.

Figure 5: Escher, Maurits. *Cycle*. 1938. Lithograph. The Palace Museum, The Hague.

Figure 6: Escher, Maurits. *Path of life*. 1958. Woodcut. The Palace Museum, The Hague.

Figure 7: Balaji, Aakarsh Vir. "Formation of a Donut from a Coffee Mug" *Veritas*, 14 Nov. 2024, www.veritasnewspaper.org/post/topology-and-the-concept-of-continuous-deformation.

Figure 8: Escher, Maurits. Mobius Strip II. 1963. Woodcut. The Palace Museum, The Hague.

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