What if I told you that in the 17th century, a single mathematical discovery acted as a time machine, effectively doubling the life expectancy of every scientist on Earth?

-"Prior to 1614, the advance in the knowledge of man was in chains because of the fatigue associated with the arithmetic process. Some of the finest minds in astronomy were slaves to their numbers and spent decades struggling in the sea of multiplications in an attempt to measure the path of a star. Time was slipping away before they could make their discoveries."

-This is where John Napier comes into the picture with his "Miraculous Rule" - The Logarithm. It can easily be stated that this mathematical breakthrough was to the mind what the telescope was to the eye. While the telescope helped Johannes Kepler peel off the layers covering the sky in order to see the dance of the planets, logarithms enabled him to see through the mathematical fog to calculate their exact orbits by transforming multiplication into:

log_b(x · y) = log_b(x) + log_b(x)

-"Not only did Napier find solutions for equations, but he spearheaded a revolution in science that rebelled against the oppression caused by big numbers. Napier basically saved Kepler and other scientists from ‘computational exhaustion’ by changing years of painstaking calculation to mere hours of work. Napier’s contribution gave science a second chance by demonstrating that some of our most profound revelations about the universe may lie not in distant observation, but in improved computation."

The basic relationship is:

$$\text{If:} \quad X = 10^Y \quad \text{then} \quad \log_{10}(X) = Y$$

X	**10^Y**	**$Y = \log_{10}(X)$**
1,000,000	**10^6**	**6**
100,000	**10^5**	**5**
10,000	**10^4**	**4**
1,000	**10^3**	**3**
100	**10^2**	**2**
10	**10^1**	**1**

1	**10^0**	**0**
0.1	**10^{-1}**	**-1**
0.01	**10^{-2}**	**-2**
0.001	**10^{-3}**	**-3**
0.0001	**10^{-4}**	**-4**
0.00001	**10^{-5}**	**-5**
0.000001	**10^{-6}**	**-6**

Notes:

1. **You can't take the logarithm of a negative number or of zero.**

2. **The logarithm of a positive number may be negative or zero.**

3. **Different books and Tables use different notations: log(X) without the subscript may mean either $\log_{10}(X)$ or $\log_e(X)$.**

4. **The natural logarithm of a number is always 2.30258...... times the common logarithm of that number. ln (10) = 2.30258.....**

If you are given that the common logarithm is an integer or a negative integer, you know that the number is 10 raised to that power:

If: $\log_{10}(X) = 2$, $X = 10^2$ or 100 .

If: $\log_{10}(X) = -3$, $X = 10^{-3}$ or 1/1,000.

If you are given $\log_{10}(X) = 1.9156$, you know that

$X = 10^{1.9156}$,

and is between 10^1 (10) and 10^2 (100) - probably closer to 100. You can also say that it is a number between 1 and 10 multiplied by 10^1 (#.##...). If you put 1.9156 into the right (Y) side of the calculator above, and click on 10^Y , the result is 82.3379..... , as we saw earlier.

If you are given $\log_{10}(X) = -2.56$, you know that

$X = 10^{-2.56}$,

and is between 10^{-2} (0.01) and 10^{-3} (0.001). You can also say that it is a number between 1 and 10 multiplied by 10^{-3} (0.00###...). The calculator will show you that $X = 2.754 \times 10^{-3}$ (2.754...E-3) or 0.002754.

Mathematical Operations with Logarithms

Logarithms change mathematical operations:

Multiplication is replaced by Addition:

$$\log(X*Y) = \log(X) + \log(Y) ,$$

Division is replaced by Subtraction:

$$\log(X/Y) = \log(X) - \log(Y) .$$

Roots and Powers become:

$$\log(X^n) = n \log(X) \quad , \quad \log(X^{1/n}) = (1/n) \log(X) \; .$$

(Bertoline, n.d)

A logarithm is a function that does all this work for you. We define one type of logarithm (called "log base 2" and denoted $\log_2$) to be the solution to the problems I just asked. Log base 2 is defined so that

$$\log_2 c=k$$

is the solution to the problem

$$2^k \;=c$$

for any given number C. In other words, the logarithm gives the exponent as the output if you give it the exponentiation result as the input. To get all answers for the above problems, we just need to give the logarithm the exponentiation result C and it will give the right exponent k of 2. The solution to the above problems are:

$$\log_2 8 =3$$

Just like we can change the base b for the exponential function, we can also change the base b for the logarithmic function. The logarithm with base b is defined so that

$$\log_b (c) = k$$

is the solution to the problem b^k=c for any given number c and any base b.

For example, since we can calculate that 10^3=1000, we know that $\log_{10}1000$=3 ("log base 10 of 1000 is 3"). Using base 10 is fairly common. But, since in science, we typically use exponents with base e, it's even more natural to use e for the base of the logarithm. This natural logarithm is frequently denoted by ln(x), i.e.,

$$\ln(x)=\log_e x.$$

In other words, $k = \ln(c)$ (1) is the solution to the problem e^k=c (2) for any number c. Since using base e is so natural to mathematicians, they will sometimes just use the notation log x instead of ln x. However, others might use the notation log x for a logarithm base 10, i.e., as a shorthand notation for $\log_{10} x$. Because of this ambiguity, if someone uses log x without stating the base of the logarithm, you might not know what base they are implying. In that case, it's good to ask.

Basic rules for logarithms

Since taking a logarithm is the opposite of exponentiation (more precisely, the logarithmic function $\log_b x$ is the inverse function of the exponential function b^x), we can derive the basic rules for logarithms from the basic rules for exponents.

For simplicity, we'll write the rules in terms of the natural logarithm ln(x). The rules apply for any logarithm $\log_b x$, except that you have to replace any occurence of e with the new base b.

The natural log was defined by equations (1) and (2). If we plug the value of k from equation (1) into equation (2), we determine that a relationship between the natural log and the exponential function

$$e^{\ln c} = c. \qquad (3)$$

Or, if we plug in the value of c from (2) into equation (1), we'll obtain another relationship

$$\ln(e^k)=k. \qquad (4)$$

These equations simply state that e^x and $\ln x$ are inverse functions. We'll use equations (3) and (4) to derive the following rules for the logarithm.

Rule or special case	Formula
Product	$\ln(xy)=\ln(x)+\ln(y)$
Quotient	$\ln(x/y)=\ln(x)-\ln(y)$
Log of power	$\ln(x^y)=y\ln(x)$
Log of e	$\ln(e)=1$
Log of one	$\ln(1)=0$
Log reciprocal	$\ln(1/x)=-\ln(x)$

The product rule

We can use the product rule for exponentiation to derive a corresponding product rule for logarithms. Using the base b=e, the product rule for exponentials is

$$e^a \times e^b = e^{a+b}$$

for any numbers a and b. Starting with the log of the product of x and y, ln(xy), we'll use equation (3) (with c=xy) to write: $e^{\ln(xy)} = xy.$

Then, we'll use equation (3) two more times (with c=x and with c=y) to write xy in terms of ln(x) and ln(y),

$$e^{\ln(xy)} = xy = e^{\ln(x)} e^{\ln(y)}.$$

Lastly, we use the product rule for exponents with a = ln(x) and b = ln(y) to conclude that

$$e^{\ln(xy)} = e^{\ln(x)} e^{\ln(y)} = e^{\ln(x) + \ln(y)} .$$

When we take the logarithm of both sides of $e^{\ln(xy)} = e^{\ln(x) + \ln(y)}$, we obtain

$$\ln(e^{\ln(xy)}) = \ln(e^{\ln(x)+\ln(y)}).$$

The logarithms and exponentials cancel each other out (equation (4)), giving our product rule for logarithms,

$$\ln(xy) = \ln(x) + \ln(y).$$

The quotient rule

The quotient rule for logarithms follows from the quotient rule for exponentiation,

$$e^{a-b} = \frac{ea}{eb}$$

in the same way.

Starting with c=x/y in equation (3) and applying it again with c=x and c=y, we can calculate that

$$e^{\ln(x/y)} = \frac{X}{Y} = \frac{e\ln(x)}{e\ln(Y)} = e^{\ln(x) - \ln(y)}$$

where in the last step we used the quotient rule for exponentation with $a=\ln(x)$ and $b=\ln(y)$. Since $e^{\ln(x/y)} = e^{\ln(x) - \ln(y)}$, we can conclude that the quotient rule for logarithms is

$$\ln(x/y)=\ln(x)-\ln(y).$$

(This last step could follow from, for example, taking logarithms of both sides of $e^{\ln(x/y)} = e^{\ln(x)-\ln(y)}$ like we did in the last step for the product rule.)

Log of a power

To obtain the rule for the log of a power, we start with the rule for power of a power,

$$(e^a)^b=e^{ab}.$$

Starting with $c=x_y$ in equation (3) and applying it again, this time just once more with $c=x$, we can calculate that

$$e^{\ln(x^y)} = x^y = (e^{\ln(x)})^y = e^{y\ln(x)}$$

where in the last step we used the power of a power rule for $a=\ln(x)$ and $b=y$. From $e^{\ln(xy)} = e^{y\ln(x)}$, we can conclude that $\ln(x^y)=y\ln(x)$,

which is the rule for the log of a power.

Log of e

The formula for the log of e comes from the formula for the power of one,

$$e^1=e.$$

Just take the logarithm of both sides of this equation and use equation (4) to conclude that

$$\ln(e)=1.$$

Log of one

The formula for the log of one comes from the formula for the power of zero,

$$e^0=1.$$

Just take the logarithm of both sides of this equation and use equation (4) to conclude that

$$\ln(1)=0.$$

Log of reciprocal

The rule for the log of a reciprocal follows from the rule for the power of negative one

$$x^{-1} = \frac{1}{x}$$

and the above rule for the log of a power. Just substitute y = −1 into the log of the power rule, and you have that

$$\ln(1/x) = -\ln(x).$$

) Nykamp ،n.d(.

-Logarithms are not just a product of the 17th century but also the silent powerhouses of today’s computational complexities and theories of information. In the world of Big Data, the use of logarithms is as crucial as ever in handling enormous numeric databases. The ability of logarithms to convert a mathematical product into a sum, a basic principle, allows modern-day computing machinery to handle the problem of scaling of exponentials. One of the areas in which this concept is extensively used is in Artificial Intelligence, where 'Log Likelihood' functions help ease complicated probability computations.

The Future of Logarithmic Thought

"It is the logarithm that finally provided the connection that helped the likes of Napier and Kepler transcend the tedious era of manual calculations and step into the fascinating age of discoveries in the sky. Nevertheless, the importance of the logarithm goes beyond this, for it remains the common language through which all the complexities of our world are distilled and made sense of in neat systems. Whether in the minuscule world of quantum physics or in the vast world of astronomy, the logarithm illustrates to us the true genius of humans in simplifying complexities. In the realm of artificial intelligence, the logarithm remains the map to uncharted territory."

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(Bertoline, n.d)

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