Striking a Mystery

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1 Introduction

Imagine you are listening to two drums being struck. By ear alone, can you work out their shapes? Tell if they're the same? This is the question that Mark Kac poses in his 1966 paper, "Can You Hear the Shape of a Drum?"(Kac 1966) and the question which so deeply intrigued me when I first heard it, as I had always been slightly rubbish at music and deeply admired those skilful enough to recognise a note by ear alone. For me, this question seemed to offer an insight, perhaps even a mathematical excuse for my own inability, into this phenomenon. This question was also fascinating to me in that it is seemingly very simple, stateable in only eight basic words, and yet it was only answered 34 years ago, going unanswered for 26 years after it was posed.

2 Eigen What Nows?

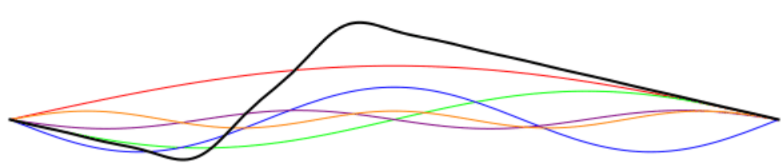


Figure 1: The displacement of a string at a point in time (black) and its decomposition as the linear combination of 5 modes consisting of the first 5 harmonics (other colours)

As is often helpful, let's start with a simpler related question, which provides some insight into this problem, and in our case this will be a 1-D string. As the only notion of shape for a (simply connected) 1-D domain is length, our question for now is "Can you hear the length of a string?" As per Figure 1, the displacement of a plucked string might appear complex and initially quite random, but is actually a linear superposition (i.e. only multiplied by constant scalars and added together) of sinusoidal harmonics. These harmonics are the frequencies at which the string resonates, and those familiar with AS level physics might recall that these are the standing waves (waves which do not appear to move spatially and only oscillate in place) formed by the string.

In these strings, the frequency of the lowest frequency standing wave that forms is called the fundamental frequency, and all other standing waves that form are integer multiples of this frequency.

The actual oscillation of the string, which produces a sound, is a superposition of these harmonics (with the exact amplitudes of these harmonics giving the note its texture and distinguishing a harp from a piano, for example). Marin Mersenne discovered in 1636 that the fundamental frequency was inversely proportional to the length, L, and promulgated the equation

$$f_0 = \frac{1}{2L}\sqrt{\frac{T}{\mu}}$$

where T and μ are the tension and linear mass density of the string. From this, we can see that the fundamental frequency alone is enough to determine the length of the string (provided the other constants are known), and therefore one can "hear the length of a string" by hearing the lowest frequency pure tone in a note and using the formula above, rearranged for L,

$$L = \frac{1}{2f_0}\sqrt{\frac{T}{\mu}}$$

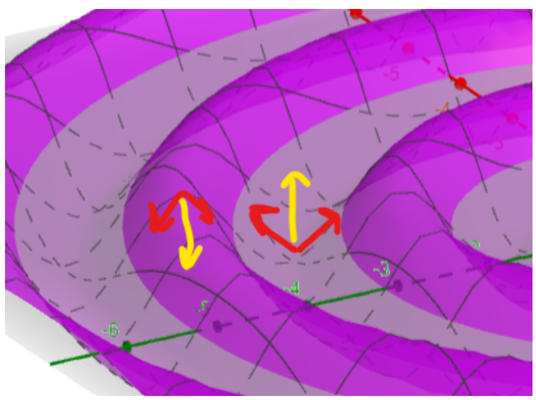


Figure 2: The forces on a peak and trough from their surroundings in red (the points beside them horizontally contribute no vertical force), note that lateral forces cancel and the resultant force is shown in yellow. Made by the Author

Extending this to our drum[1], we need some way of working out the frequencies at which standing waves can form on the drum's surface and in order to do this, we need to know how waves propagate through a two dimensional surface. The elastic membrane of a drum lends us some physical intuition because every point will have some tension acting upon it by its surrounding points, and will therefore be pulled down if it is higher than most of them, and vice versa, and thus we should expect a point to accelerate toward the average of the points around it, as shown in Figure 2. This is described mathematically by the wave equation, first written in 1746,

$$\frac{\partial^2 u}{\partial t^2} = v^2\,\nabla^2 u$$

Whilst this equation might seem slightly baffling to those unfamiliar with the notation in which Partial Differential Equations are described, it is essentially saying exactly what was described above. u here is the function $u(x, y, t)$ and represents the vertical displacement of the drum at a position at a moment in time, whilst $\frac{\partial^2 u}{\partial t^2}$ refers to the partial second derivative of this displacement

[1]For our purposes, a thin membrane held fixed at the edges on which no drag acts.

function with respect to time, and can be thought of as simply the acceleration of a point like the ones we were thinking about earlier. Meanwhile, $\nabla^2 u$ is the Laplacian operator, which is the sum of all the spatial second derivatives (or equivalently the divergence of the gradient, showing how much a point 'protrudes' from those around it), and represents how different the displacement of a point is from the average of those around it (equivalent to the yellow arrow in Figure 2). v^2 is just the square of the speed of the membrane, and is unimportant for our purposes (we can set it to whatever we want through a cunning choice of units). Again, this equation is just saying that every point accelerates towards the average of the points around it (i.e troughs will move up, peaks will move down).

The function u is defined on a 2-D domain $\Omega \subset \mathbb{R}^2$, which represents the membrane of the drum and is the shape we are trying to hear. u is also set to zero at the edges, $u|_{\partial\Omega} = 0$ because a drum is held fixed at its boundary.

Because we are looking for functions u which are purely harmonic, (i.e. oscillate periodically in time and therefore produce only one frequency), we can let the us we care about be written as the product of a time invariant $U(x, y)$ (sort of the max amplitude of each point except it can be negative as well) and, with ω being the angular frequency of this standing wave, a sinusoidal term $e^{-i\omega t}$ (we take the real part to represent the physical displacement, which for those unfamiliar with imaginary numbers simply gives us $\cos(\omega t)$), giving

$$u(t, x, y) = U(x, y)\, e^{-i\omega t}$$

Substituting this form back into the wave equation and simplifying (using the fact that $U(x, y)$ does not depend on time whereas the exponential term is purely temporal), we end up with the equation

$$-\frac{\omega^2}{v^2} U = \nabla^2 U$$

Using some more of the trusty AS level waves topic, we can see that $\frac{\omega^2}{v^2}$ can be simplified to k^2, where k is the angular wavenumber (number of radians per unit distance) giving,

$$-k^2 U = \nabla^2 U$$

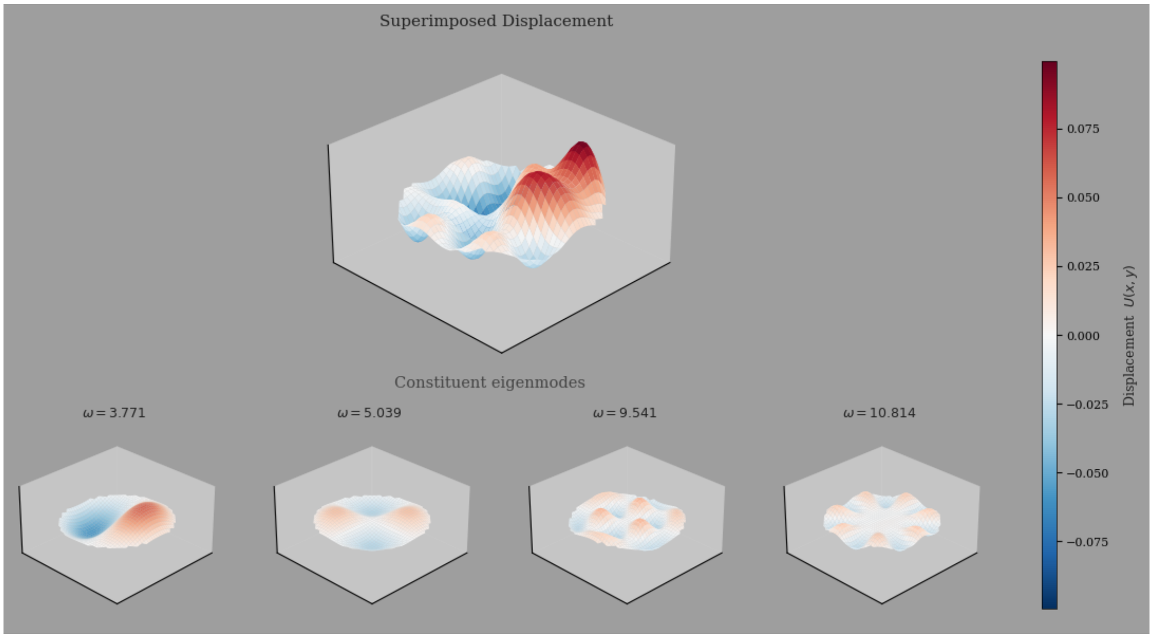


Figure 3: A complex oscillation of a circular drum and its decomposition into multiple eigenmodes of various eigenfrequencies. Made by the Author using Scipy and Matplotlib

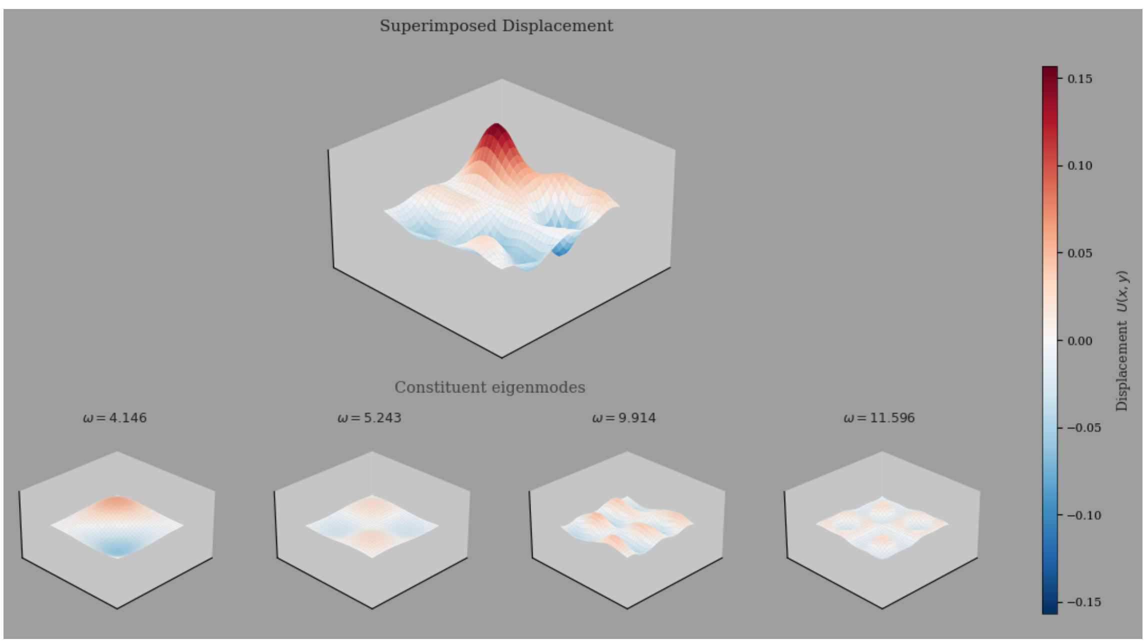


Figure 4: A complex oscillation of a square drum and its decomposition into multiple eigenmodes of various eigenfrequencies. Made by the Author using Scipy and Matplotlib

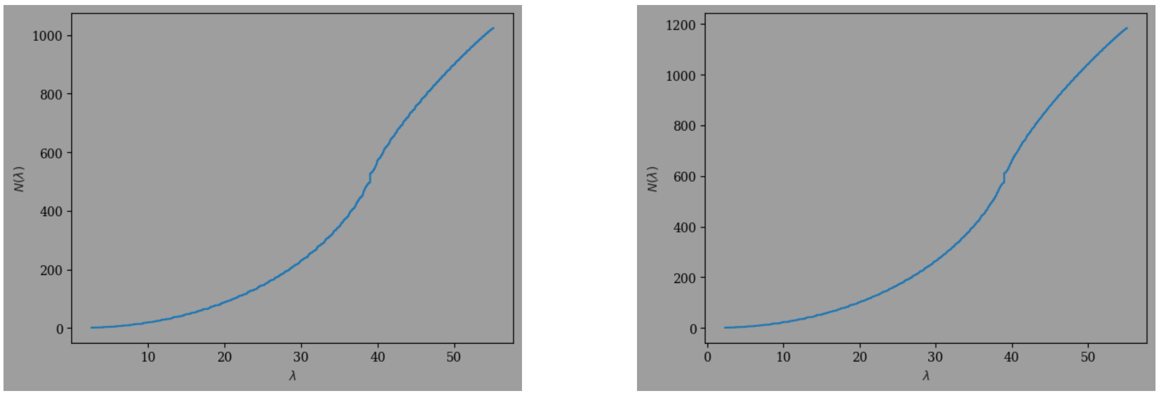
This equation, named after its discoverer Hermann von Helmholtz (who studied it in 1860), is stating that for every angular wavenumber for which there exists a standing wave (and therefore a pure tone at the corresponding frequency), there must be a function $U(x, y)$ which when the Laplace operator is applied to it, is merely scaled (by $-k^2$), which will always be a negative amount. This might seem familiar to the problem in linear algebra of finding a vector that when a given matrix (which is essentially an operator) is applied to it, is merely scaled and it is so similar in fact that we adopt similar terminology. The function $U(x, y)$ is referred to as the eigenfunction (analogous to the eigenvector in the linear algebra equivalent) whilst the amount by which it is scaled, $-k^2$ is referred to as the eigenvalue (or in this case eigenfrequency). A given domain will have an infinite number of eigenvalues (just like a mathematical plucked string has an infinite number of harmonics, although they get pretty quiet $\dots$), and the complete list of the eigenvalues is referred to as the spectrum of the domain. Numerically approximated eigenfunctions, and their corresponding eigenfrequencies, as well as how they superimpose to make complex oscillations, are shown in Figures 3 and 4.

Our original question, "Can you hear the shape of a drum?", is therefore equivalent to (the slightly more complicated) "Does the spectrum of the Laplace operator on a given domain uniquely determine its geometry?".

We now know what the question that needs answering actually is, however it is really non-obvious how it can be answered. If it were to be answered negatively (spoiler alert), it would suffice to find two isospectral domains that are non-congruent (in other words, two differently shaped drums that produce the same notes), however how to go about this is also not clear. Firstly we have the issue that in general one cannot find an analytical solution for the eigenfunction or eigenvalues and must instead utilise numerical approximations on a computer (as I have been doing to make the breathtaking images) which are, by necessity, of limited precision, furthermore, there are an infinite number of eigenvalues and a computer obviously cannot calculate them all. Therefore something rather ingenious must be used just to prove two domains are isospectral, let alone find them. It is unclear to me what a positive proof might have looked like (if there was one), but it's alright because there isn't one.

3 What We Can Tell

While I have rather jumped ahead in that last paragraph and rather gave the game away, it is not completely obvious that the spectrum cannot completely determine the geometry, as it does reveal a lot of information about it.



(a) $N(\lambda)$ vs λ for a square domain

(b) $N(\lambda)$ vs λ for a circular domain

Figure 5: A comparison of the behaviour of the number of eigenvalues less than a given eigenfrequency for two domains, revealing the similarity of asymptotic behaviour which we would expect given their similar areas and perimeters. Made by the Author.

At the beginning of the 20th century, Hermann Weyl had produced an eponymous law (Weyl 1911), utilising Hilbert's relatively new integral equations, which states that the asymptotic behaviour of the frequency eigenvalues can determine the area of the domain. As $\lambda \to \infty$,

$$N(\lambda) \sim \frac{|\Omega|}{4\pi}\lambda$$

Where $|\Omega|$ is the area of the drum.

Whilst this is sort of stretching what it means to "hear the drum" because it's somewhat unlikely that even the most prodigious musician could hear the millionth eigenvalue of a drum (not least because the pesky physical realities of air resistance and thickness of the drum would certainly have completely damped it out), this is equivalent to our earlier 1-D case. After all area is the 2-D equivalent of 1-D length (in the sense of area being the measure of $\mathbb{R}^2$) and in this sense a good musician could hear the area of a tambourine in the same way they could hear the length of a violin string (modulating the length of the string is, after all, how you play one), with the slight caveat that they must know the relevant material properties of the instrument in advance.

In fact, it gets even more promising, because you can actually hear both the perimeter (the length of the circumference of the drum) as well as the connectedness of the drum as well, again as the asymptotic behaviour of the spectra. Kac himself conjectures in his 1966 paper (Kac 1966) that, with L as the circumference and r the number of holes in the drum,

$$\sum_{n=1}^{\infty} e^{-\lambda_n t} \sim \frac{|\Omega|}{2\pi t} - \frac{L}{4}\frac{1}{\sqrt{2\pi t}} + (1-r)\frac{1}{6}$$

With $\sim$ just denoting asymptotic equality as $t \to 0^+$. Whilst the derivation and proof of this result is beyond the scope of the essay, it reveals that an enormous amount of information about the geometry of the drum can in fact be determined by its spectra. This is of great practical use, as will be discussed in a later section, but it is still unclear whether the complete geometry might be determinable.

Going back to my grinch-like spoilers, this at least means that any different drums with identical spectra must have the same area, perimeter and number of holes, which is a very limiting constraint.

4 What We Cannot Tell

Alas, as I have been alluding to throughout, in the words of its provers, "One cannot hear the shape of a drum". In order to prove this, in 1992 Carolyn Gordon, David L. Webb, and Scott Wolpert (Gordon, Webb, and Wolpert 1992), utilised a theorem by a man called Toshikazu Sunada (Sunada 1985) to prove that shapes with certain symmetries have identical algebraic properties.

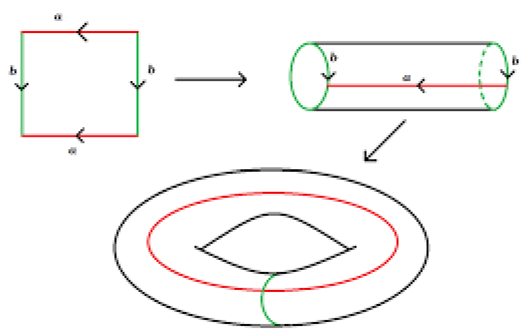
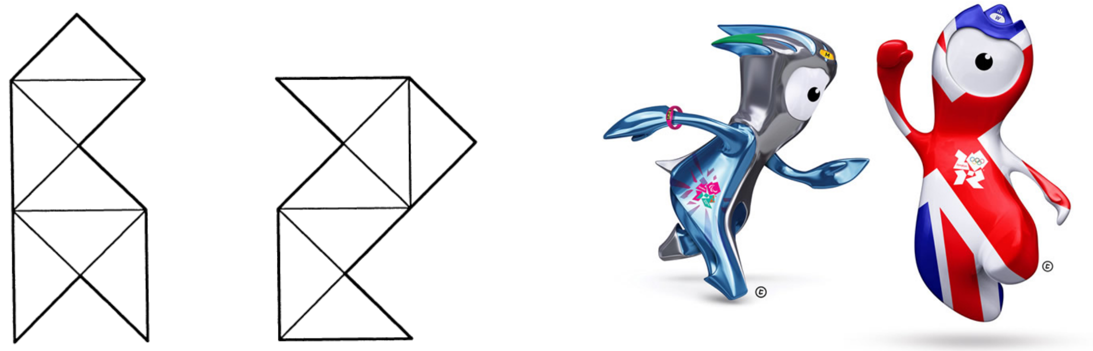


Figure 6: A diagram showing the formation of a Quotient Space. Note the symmetrical points being glued together to form a new shape. Source: ResearchGate

The ingenious mechanism of their proof relied on group theory, which studies symmetries (actions like reflecting and rotating which leave an object unchanged). Sunada's theorem stated that by taking a space which has many of these symmetries (the complete collection of which constitute the symmetry group G) and creating two subgroups, $H_1, H_2 \subset G$ and using them to form two quotient spaces (a new space formed by gluing together points considered identical by the symmetries in the utilised subgroup, as per Figure 6), under certain conditions (namely that H_1 and H_2 are almost conjugate, i.e. each subgroup has the same number of each type of symmetry), the spectra of these quotient spaces are the same. In simpler words, by folding and gluing a highly symmetrical starting space in two very similar (but not identical) ways, such that each 'type' of fold occurs the same number of times in each way, you get two shapes that are not identical in shape, but have the same spectra.



(a) The Drums (b) The Mascots

Figure 7: A comparison of the non-isometric isospectral domains and their terrifying doppelgangers. Source: UCDavis and the London Olympics

Using this and making cunning choices for the original (highly symmetrical) shape and H_1 and H_2, they generated two differently shaped drums (which I thought looked uncomfortably like the terrifying 2012 Olympic mascots that had haunted my early childhood), which demonstrated that the answer to Kac's 1966 question was definitively no. The symmetries of the two shapes are quite clear however, and you can see how it might have been made by using a series of folds involving a right-angled triangle, while also noting the shape's identical perimeters and areas (along with their lack of holes) as we found was required in the previous section.

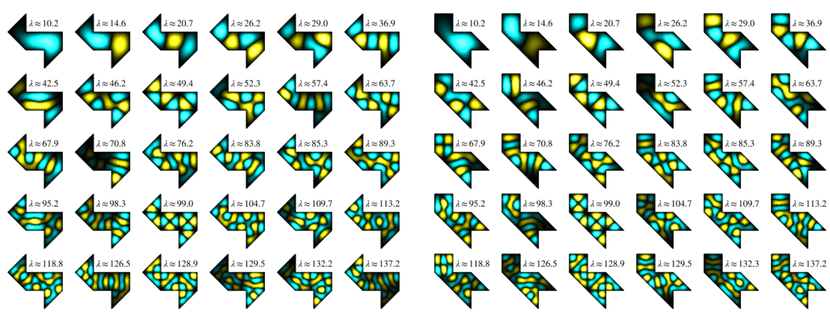


Figure 8: A numerical demonstration of the eigenvalues of two isospectral but non-isometric domains, confirming the validity of the solution (to numerical precision)

These solutions can be verified with numerical computation, as shown in Figure 8 (although this is not how they were proved).

This means, more generally, that if I blindfolded you and had you tap dinner plates with spoons you could work out their relative areas, and could generally tell dinner plates apart from mugs, but there does exist a specific, likely bizarrely shaped geometry of mug that would be indistinguishable from a plate.

Their method was extended by Chapman (Chapman 1995) to generate many such isospectral drums, however if the domains considered are restricted to "certain convex planar regions with analytic [boundaries]", then it has been proven that you can actually determine the geometry from the eigenvalues (Zelditch 1998).

5 Real World Applications

This question, far from being only a curiosity (not that it isn't absolutely fascinating in its own right, however this was an applied question to begin with. . .), turns out to have extensive practical relevance. The field of determining the geometry of something from its spectrum (known as Inverse Spectral Geometry), has applications across science and engineering. Although most of these applications are in three dimensions, the general result, that the spectrum does not uniquely determine this geometry, still applies whilst the topics we have covered on the way, such as spectral analysis, are incredibly useful.

Take, for example, Non-Destructive Testing in structural Engineering, where ultrasound waves are sent through structural beams, and the echoes and resonance these waves experience are used to identify potential breaks and defects. Meanwhile, in computer vision, the Laplacian spectrum of a 3D mesh serves as a form of fingerprint, because it doesn't change much even after bending or deformation, meaning the same thing can be recognised regardless of its arrangement (e.g. a curled-up cat). Furthermore, in the field of spectroscopy, chemists study the geometry of molecules by observing the eigenfrequencies of electron energy levels, "hearing" the shape of a chemical bond. In medical imaging, techniques like Magnetic Resonance Elastography (MRE) vibrate our soft tissues, listening for stiffness that might indicate a tumour.

All of these uses rely on the maths we have explored today, and are bound by the fact that geometry cannot be completely determined by sound (i.e. you cannot hear everything, no matter how good the setup and microphone), despite the trove of information that can in fact be supplied to us by the spectrum.

6 Conclusion

This seemingly inauspicious question, which I found incredibly interesting despite my complete lack of musical talent, took me on a journey through multivariable calculus, partial differential equations and even a fleeting dip into group theory, whilst also tracing a historical path which intersects with some of the best mathematical minds and physicists of the 20th and 19th centuries, and I hope you have enjoyed it as much as I did.

We have learnt that the sound of a drum can tell you its area, the length of its rim, and how many holes it has, but crucially it cannot tell you its shape. Two different drums, produced by mathematical origami, can sound the exact same. This does somewhat excuse my lack of auditory skill, although not for string instruments, and reveals the limits of listening.

While we have, via algebra and group theory, landed upon a negative result which rules out entire classes of reconstruction algorithms for geometries based on spectra, the field is by no means closed (another atrocious pun intended), as the exact cases and methods by which domains can be determined by their spectra remain to be found, leaving an uncharted landscape for further findings to be 'sounded out'.

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