Making Money Mathematically

By: Oliver Turner

Note: This essay is about a game used to gamble money, however I do not recommend gambling in any form, it is a quick way to lose money.

Introduction

Hello fellow maths enthusiasts, I hope you enjoy the read! In this essay I will talk about a topic that is heavily reliant on maths, but relies more on problem solving than formulas and equations.

You may know of a card game known as “Twenty-One”, also referred to as “Blackjack” when played with money on the line. It’s a game of chance: Get dealt (usually) 2 cards and hope you’re closer to a “value” of 21 than the dealer (who is also dealt 2 cards), without going over 21. **To many people, it’s a game of pure luck**, of pure chance… However, we are mathematicians, “chance” is just another way maths snakes its way into everyday life, in this case: probability and statistics.

The Game

The concept of the game is simple, but first let’s define some of the terms I’ll be using in this essay.

“Value” is a number assigned to each type (not suit) of card in a standard deck of playing cards. The number cards all have a “Value” equal to the number on that card, simple! Picture cards (Jacks, Queens, Kings) have a value of 10 (once again, easy enough). The ace is a bit special, able to count as 11 or 1 depending on which is better for you.

“N” is the number of decks used.

“#” is shorthand for number of/number of cards with.

“V” is shorthand for value.

“C” is shorthand for cards.

“P(x)” is shorthand for probability of x, where x is an event/situation.

The game plays quite simply, the dealer deals every player 2 cards face up, and while doing so also deals themselves with one face up card, one face down card. Your goal is to have your cards’ total value closer to 21 than the dealer’s 2 cards, without going over 21 (if you go over 21 you immediately lose). If you’re closer – you win (the hand)! If you get exactly 21 –

you win (the hand)! You can also choose after seeing your 2 cards to “hit”, where you get dealt another card. You can do this as many times as you wish (until you choose to stop, or you go over 21 in value). The dealer must “hit” if the value of their cards is less than or equal to 17. Different ways of winning can lead to different amounts of money won.

The game is usually played with multiple decks shuffled together. We will refer to the number of decks used as “N”.

The Maths

A deck of cards always has 52 cards, 13 of each suit, in each suit the cards numbered from 2-10, a jack/queen/king, and an Ace. This means that if we were to take one card from a single deck of cards, each possible value has a probability of being drawn. For each value 2-9 that chance is 4/52, which simplifies to 1/13. The value of 10 has a chance of 16/52, or 4/13. A card of value 11 has a chance of 4/52, or 1/13 (same as the cards of value 2-9). This is found with the simple formula:

$$P(V) = \frac{\#V}{\#C}$$

This also means that the chance for each value is the same no matter the value of N: (in this hypothetical)

$$P(V) = \frac{N\#V_1}{N\#C_1} = \frac{\#V_1}{\#C_1}$$, (Where subscript 1 means when N=1)

If you wanted to find the chance of getting a certain value if instead you were given 2 cards from the same deck, the calculation is harder, as you must find all the combinations that give that value, and then sum the chances for those combinations. See the following equation:

$$P(V) = \sum\left(\frac{\#V_a}{\#C_1} \times \frac{\#V_b}{\#C_1 - 1}\right) = \sum \frac{\#V_a \#V_b}{\#C_1(\#C_1 - 1)}$$, (Where subscript a/b means “of a/b”, where a/b are Values that sum to V)

This is clearly a lengthier calculation, but with the help of computers/scripts this is also not very difficult. With N decks where N doesn’t equal 1, “$\#V_{a/b}$” can be slightly harder to work out, but it’s quite self-explanatory.

From here we can work out the probability of a player winning this game (when both you and the dealer are dealt strictly 2 cards for now) by finding the probability of the player getting any total value from their 2 cards with a range of values from 4 (two cards of value 2) to 21. Do the same for the dealer, and then when everything is calculated you can figure out how often you’d beat the dealer/win the hand. (note that the number of cards will change as cards are dealt in a hand). Then, consider the chance of various values when including the possibility to “hit” (get another card) and when the dealer must “hit”. This changes the chances slightly, and is calculated with a similar summation to above, with more possibilities. (this essay is too short to cover the probabilities of 0/1/2/3/etc hits in detail, but there's many sources about this online)

Overall Probabilities

(This next part assumes "perfect play", which is where the player always makes the decision which has the best odds of a win statistically, almost like how in checkers you can make the best move every time if you know the situations and theory behind the game.)

The percentage of games won over a lot of hands (with perfect play) averages out to a 42% chance of winning a hand. However, this ends up leading to a 99.5% "return to player" (which is how much money a player can expect to get back – with 100% being they end with the exact amount of money they start with). This sounds good, but what it means is that per £100 you have, you leave with £99.50 on average each hand. Therefore, the casino gets 50p per £100 you wager (we can say the casino has a 0.5% edge). On average, this means you'll lose more each hand you play (on average).

This percentage seems quite good at a first glance. "Only Losing 50p when you can make thousands?" - this is a trap which catches many people out. Remember: you will lose money with these odds. The correct way to think about this probability is that a casino will take 0.5% of your money over and over. Therefore, as a person with slightly more mathematical knowledge we can go back to the idea of blackjack being "pure luck" (from the start of this essay) and confidently say: "It's not luck, it's a unique way to lose money". Of course, luck plays a factor, but over time luck always has less of an impact. (look at the proportion of heads/tails when flipping 2 coins compared to flipping 10,000 coins).

So that's that, we found out with some simple probabilities why gambling is bad.

Unless?

There's a reason I brought up quite early the idea of multiple decks – the more decks there are, the more someone quite numerate can change the odds into their own favour.

How? Card Counting.

Note: Card Counting is in no way illegal, it is a way to think more critically about applied statistics while playing a game. Card Counting is not cheating in the same way that thinking too much about chess while playing chess is not considered cheating. (what a silly concept!)

When there are multiple decks there are more cards. When there are N many decks, there are N times the number of each card as there is with just one deck. Therefore, there are also N times the total number of cards. Some of you may be confused though, because earlier I showed that this should have no effect on the probability of drawing a certain card/value of card as the N cancels out.

I may have failed to mention that if the cards aren't replaced in the decks, then suddenly things change very quickly.

The situation before was drawing a card, and replacing it back in the decks, leading to constant probabilities (per hand). So how does not replacing the cards change the game so drastically?

How and Why it works

let's categorise all the cards with a value of 10 or 11 (10s, Jacks, Queens, Kings, Aces) as "high cards", and an equal number of the cards with the lowest values (2s, 3s, 4s, 5s, 6s) as "low cards", all the other cards aren't important (7s, 8s, 9s). In order to have a higher chance of winning a hand, we want more high cards than low cards, so that we are on average going to be closer to a value 21. This means that we can create a system to remember the ratio of high cards drawn (no longer in the decks) to low cards drawn called the "count". A higher number of low cards drawn compared to high cards drawn means that in the decks there will be more high cards than low cards, leading to a higher chance of high cards in future hands. Since the goal is to compare remaining high cards in the deck to the remaining low cards in the deck, the count is entirely relative. If a low card is drawn, that's one less low card in the decks, so there's relatively more high cards in the decks. The opposite is true when a high card is drawn, and this is also why the other cards (7s, 8s, 9s) aren't important to us. When a low card is drawn, the count therefore increases by 1, when a high card is drawn the count must decrease by 1 (and any other cards add 0 to the count, aka do nothing to the count). This allows us to quantify in a single simple number our likelihood to get high cards – not as a percentage, but as a ratio. A positive count means there's a higher chance of getting a high card compared to getting a low card. A negative count means there's a lower chance of getting a high card compared to getting a low card. A count of 0 means the probabilities are balanced.

From here what I have been referring to as the "count", will instead be called the "running count" to differentiate it from the "true count". The true count allows the relative ratio of the running count to be turned into a less abstract number that can be used to decide when we are more likely to win than the house (the casino), and therefore start betting more money. The formula for doing so is very simple:

$$True\ Count = \frac{Running\ Count}{Number\ of\ Remaining\ Decks\ (to\ the\ nearest\ 0.5\ Decks)}$$

This allows a card counter to have a frame of reference for what is a good count no matter how many decks are in play. The approximate change in "edge" (who is more likely to win) is a surprising 0.5% per 1 True Count. When the True Count is 0 (or the player isn't card counting) the player has an edge of -0.5% (aka the casino has an edge of 0.5%), so at a True count of 1, it becomes around 50/50, and it’s safe to assume a near 100% average “Return to Player”. When the count is 2, suddenly the player gets the same edge the casino would get without card counting, and as the count increases, the edge does too, and the potential earnings increase.

This allows skilled card counters to over many hands make a slight profit, leading to “professional” card counters who use maths (specifically statistics) and numeracy skills to make lots of money. Therefore, blackjack (when card counting) is a mathematical money-making method.

(Please don’t try this at home, if you don’t know what you’re doing you can lose a lot of money, and even if you do know what you’re doing it’s very possible to get extremely unlucky and lose money anyway, so don’t gamble!)

Thank you for reading this essay about applied problem solving and logical reasoning!

Sources

https://outplayed.com/blog/what-are-the-odds-of-winning-blackjack

This webpage has got an in-depth breakdown of how blackjack can be played with “perfect play”, including many statistics on the probability of winning.

https://www.youtube.com/watch?v=rjfLuM-Pqr8

This YouTube video makes understanding how to count cards easier by visualizing it (especially from 2:27 to 3:33). It also explains the rules of blackjack simply.

https://graphics.stanford.edu/~billyc/class/vis_win0304/as2/

This assignment from a university student that is publicly shared online has got lots of information about the logic behind card counting, and graphs/data about the effect of the True Count on Win rate.

https://www.youtube.com/@stevenbridges

This YouTube channel documents the journey of a professional card counter and gave me a greater insight into the practicality of the theoretical statistics behind card counting.

https://www.heartbingo.co.uk/blog/is-card-counting-illegal/#:~:text=positive%20impact%20on...-,Card%20counting%20is%20a%20tough%20technique%20to%20master%3B%20however%2C%20if,technically%20allowed%20to%20count%20cards.

This blog discusses the legality of card counting.

https://www.venetianlasvegas.com/resort/casino/table-games/how-to-play-blackjack.html#:~:text=Players%20receive%20all%20cards%20face,and%2010s%20count%20as%2010.

This informational page from a real casino explains in more depth the rules of blackjack.