The mathematics of pursuit and interception : how to chase a moving target efficiently

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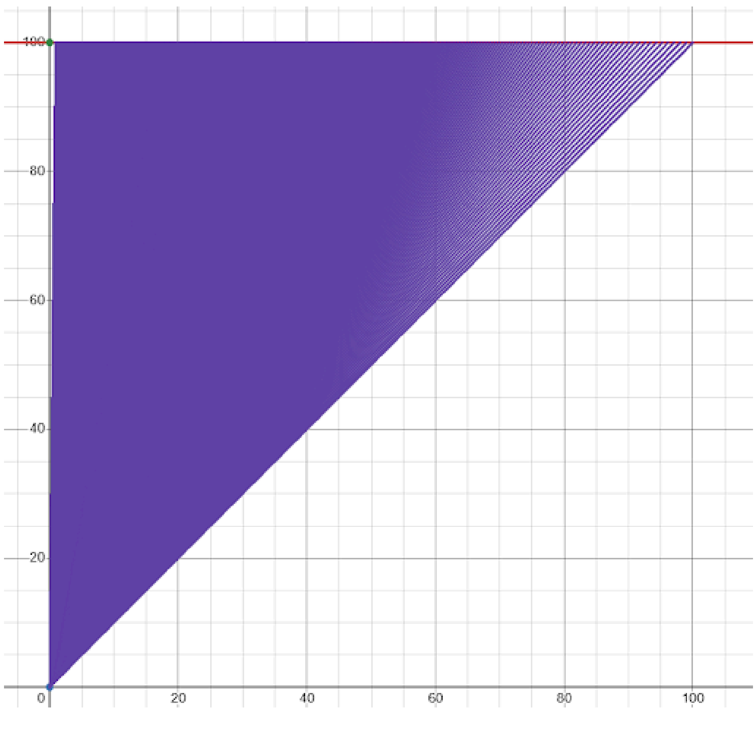
Introduction

Pursuit becomes mathematically interesting, when the chase turns into a game of prediction. It's a subconscious process - when we're trying to chase something, we try to lead it, not track where it is at the current instant.
But surely, there is a great deal of mathematics involved in this split-second decision we take. Can we formalise that rough instinct, into exact optimal paths?

Catch me if you can!

The grass rustles. Our Cheetah crouches low, his legs coiled like springs, ready to launch. The gazelle that he's long been stalking shoots off in a perpendicular direction - a race has begun.
Our feline friend accelerates at $7.5ms^{-2},$ peaking at a respectable $27ms^{-1}.$ His meal launches with an acceleration of $5ms^{-2}$, clocking $22ms^{-1}$. Only a hundred meters to cover - shouldn't be a challenge.



He could take infinitely many directions, but:

- Aim ahead by too much, and it's too much to run.
- Lead the gazelle too little, and there's hardly a point - he's behind and now running a tail chase.
- Some aren't possible.

Let's take a step back, and work from the basics.
Our cheetah starts at $(0, 0)$, and the gazelle at $(0, 100)$.
Assume our animals are like the flash. They'll **instantly** accelerate, and start off at top speeds.
Now, what must be true for an impact? They must arrive at the same point, at the same time. x and y co-ordinates must match simultaneously.

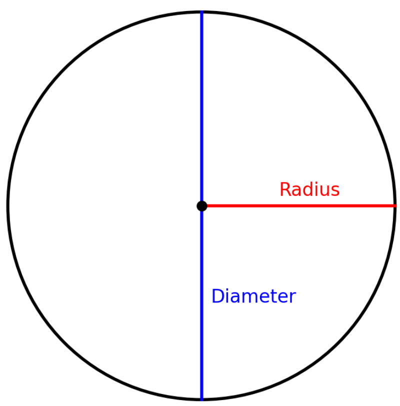
It's probably a good idea to calculate the distances they'll run, but we don't know how long they run for.
So, we can pop out the variable, t, which will represent how long it takes until impact.
The distance they run is equal to $speed \times time$. So, let's calculate it with their speeds!

Cheetah	$27t$
Gazelle	$22t$

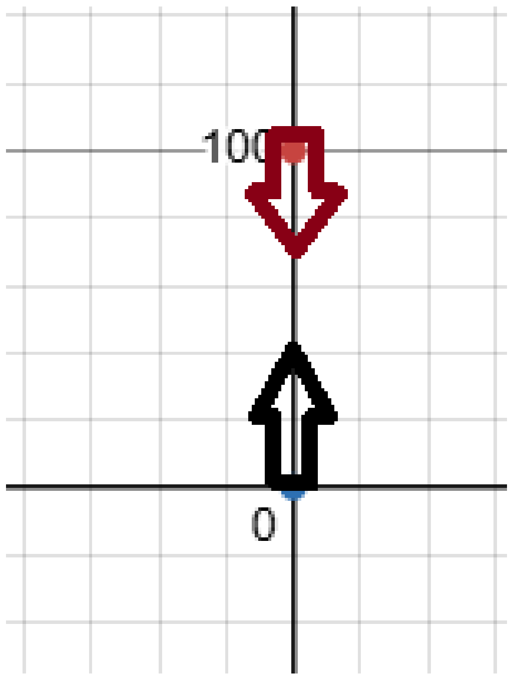
The two travel distances in a constant ratio, as $\frac{distance_{cheetah}}{distance_{gazelle}} = \frac{27t}{22t} = \frac{27}{22}$. For every 22m the gazelle runs, the cheetah runs 27m.

The ratio enables us to create a locus - a circle representing all possible interception points, also known as the '*Circle of Apollonius*'. When the ratio of distances from two fixed points is constant, the set of all possible interception points forms a circle.

Before we think about making the golden arc, we shouldn't forget the ingredients. A circle's equation is $(x-a)^2 + (y-b)^2 = r^2$. The variables here have clear origins. a and b are the x and y co-ordinates of the centre-point, respectively. r represents the radius, which is the distance from the centre to any point on the circle. We'll need to find, and halve the diameter to find r.



The two ends represent 'extremes'. The best case, where his meal gets confused and runs head-on at him (the shortest interception distance), and the worst case, where the gazelle flees directly away (the longest distance).



In the **head-on** scenario, the cheetah closes at $27ms^{-1}$, and the gazelle runs towards him, at $22ms^{-1}$. Together, the gap closes at $49ms^{-1}$. Dividing the initial 100m by this gives the time to impact.

$$27 + 22 = 49$$

$$distance = speed \ \times \ time$$

$$time = \frac{distance}{speed}$$

$$time_{min} = \frac{100}{49}s$$

Wonderful. Remember, this is the shortest interception time. So, multiply $time_{min}$ by its speed, to find the distance.

$$\begin{aligned} distance \ &= speed \ \times \ time \\ distance_{min} \ &= \frac{27}{1} \times \frac{100}{49} = \frac{2700}{49}\text{m} \end{aligned}$$

Keep it in exact form for now.

In a **tail chase**, $27ms^{-1} - 22ms^{-1}$ gives a closure speed of $5ms^{-1}$. Then, we repeat the previous steps.

$$27 \ - \ 22 \ = \ 5$$

$$\begin{aligned} time \ &= \ \frac{distance}{speed} \\ time_{max} \ &= \ \frac{100}{5} = 20s \end{aligned}$$

$$distance \ = \ speed \ \times \ time$$

$$distance_{max} \;=\; 20 \;\times\; 27 \;=\; 540\text{m}$$

As promised, the key to the magic is the **diameter**, the distance between the extremes. So, we need to find the difference between $distance_{min}$, and $distance_{max}$, and halve it to get r. Let's avoid rounding!

$$\left(540 \;-\; \frac{2700}{49}\right) = \frac{23760}{49} = diameter$$

$$radius = \frac{diameter}{2}$$
$$\frac{23760}{49} \div 2 = \frac{11880}{49}$$
$$radius = \frac{11880}{49}$$

We almost forgot the part that determines where our circle is. The **centre point**.
In both extremes, the chase happens vertically. Neither animal moves across, only up, so only the y co-ordinate changes. Our cheetah starts at $(0, 0)$ and sprints up. So the distance it runs, is the impact position's y co-ordinate!

Head-on	$\left(0, \dfrac{2700}{49}\right)$
Tail-chase	$(0, 540)$

The midpoint of these is the circle's centre point.
$average = \dfrac{sum}{amount\ of\ numbers}$, right? Let's use it on the y co-ordinate. Add the y co-ordinates, and divide by 2.

$$average = \frac{sum}{amount\ of\ numbers}$$
$$sum \;=\; \frac{540}{1} + \frac{2700}{49} = \frac{29160}{49}$$

There's 2 numbers.

$$average = \frac{29160}{49} \div 2$$
$$average = \frac{14580}{49}$$

The x co-ordinate is expectedly zero. The y co-ordinate is $\dfrac{14580}{49}$. So, the centre-point is at $\left(0,\; \dfrac{14580}{49}\right)$.
It's time for our equation to resurface! Plug in our new values.

$$(x - a)^2 + (y - b)^2 = r^2$$

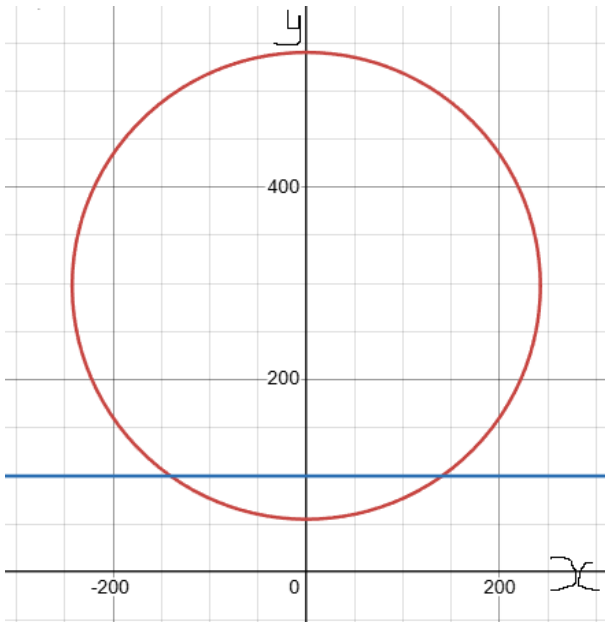
$$a = 0$$

$$b = \frac{14580}{49}$$

$$r = \frac{11880}{49}$$

$$x^2 + \left(y - \frac{14580}{49}\right)^2 = \left(\frac{11880}{49}\right)^2$$

Just a few steps away from the meal!
Remember, the gazelle is 100m away, and flees horizontally, so the path lies on the line $y = 100$. The y co-ordinate never changes.



Now, think about the circle.
Apollonius' circle represents all interception points. Surely, if we see where our gazelle's path meets the circle, we should find the point of interception!
We have the variable y, and it's equal to 100. So, we can replace y with 100 in the circle equation, and find x - how far across they meet.

$$x^2 + \left(y - \frac{14580}{49}\right)^2 = \left(\frac{11880}{49}\right)^2$$
$$x^2 + \left(100 - \frac{14580}{49}\right)^2 = \left(\frac{11880}{49}\right)^2$$

Don't calculate $100 - \frac{14580}{49}$; keep it like that to avoid ugly fractions. Now, isolate x^2.

$$x^2 = \left(\frac{11880}{49}\right)^2 - \left(100 - \frac{14580}{49}\right)^2$$

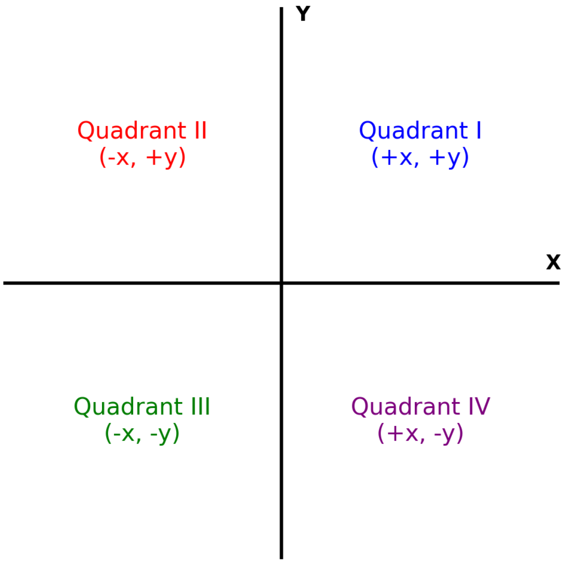
Square root both sides to get x!

$$x = \pm\sqrt{\left(\frac{11880}{49}\right)^2 - \left(100 - \frac{14580}{49}\right)^2} \rightarrow x = \pm 140.55\ (2dp)$$

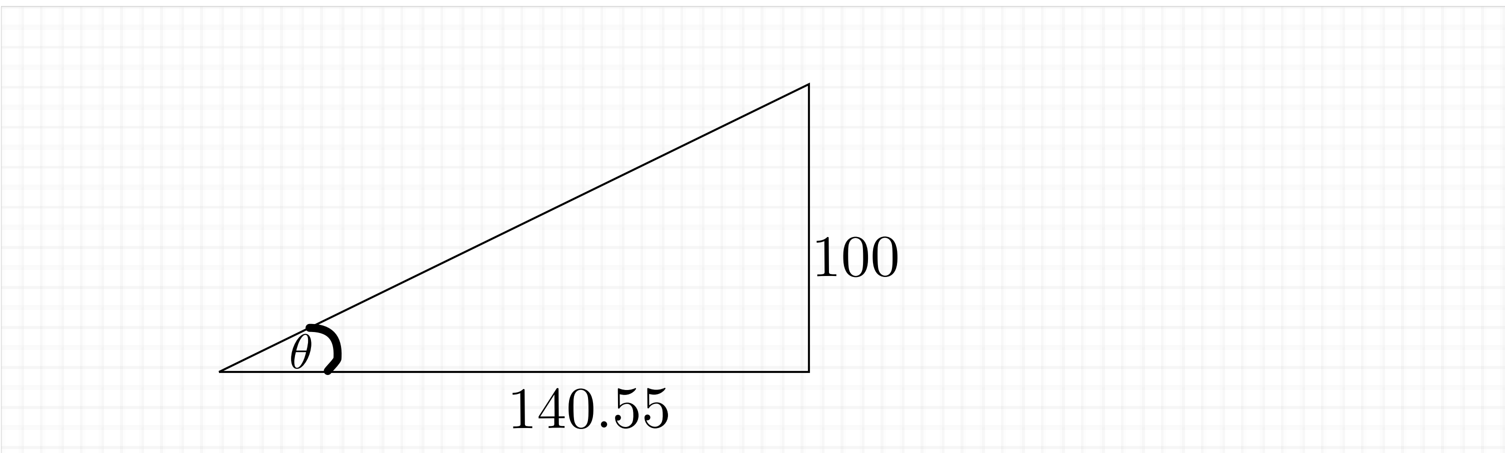
Surprising... Two points? That's odd. We should meet the gazelle only once. Let's ponder.

Assume the gazelle sets off to the right: our cheetah will meet it in the 1st quadrant (top right), where all x values are positive, so $+140.55$ works!
Or, the left: it's intercepted in the 2nd quadrant (top left), where all x values are negative. -140.55 works.

So, both solutions are valid; it just depends on the gazelle's mood!



To calculate θ (the **angle of attack**), draw a right-angled triangle. We know how far up, and across the interception point is.



We have the opposite and adjacent sides. The first thing to come to mind is $\tan(\theta)$:

$$\tan(\theta) = \frac{opposite}{adjacent}$$

Now, sub in those lengths.

$$\tan(\theta) = \frac{100}{140.55}$$

Do the inverse to unravel θ!

$$\theta = \tan^{-1}\left(\frac{100}{140.55}\right)$$

$$\theta = 35.43°$$

Disclaimer: all angles are in degrees, to the horizontal (x axis)!

Tame the kitten, and tell it to run at roughly $35.43°$ to the horizontal, and it'll be munching lunch in seconds.

Not so fast...

v : speed $\left(\text{m}s^{-1}\right)$
a : acceleration $\left(\text{m}s^{-2}\right)$
t : time (s)
s : distance $\left(\text{m}\right)$

In reality, they aren't running at top speeds off the bat. So, let's consider **acceleration**.

	Cheetah	Gazelle
Speed $\left(\text{m}s^{-1}\right)$	27	22
Acceleration $\left(\text{m}s^{-2}\right)$	7.5	5

We won't use our earlier circle theorem, as the ratio of distances isn't fixed. Acceleration is involved, so speeds aren't constant.
This dilemma calls for a **piecewise function**! A piecewise function is one where the equation to calculate the output, depends on the **interval** the input falls into. Here, the equation to calculate distance, depends on time.

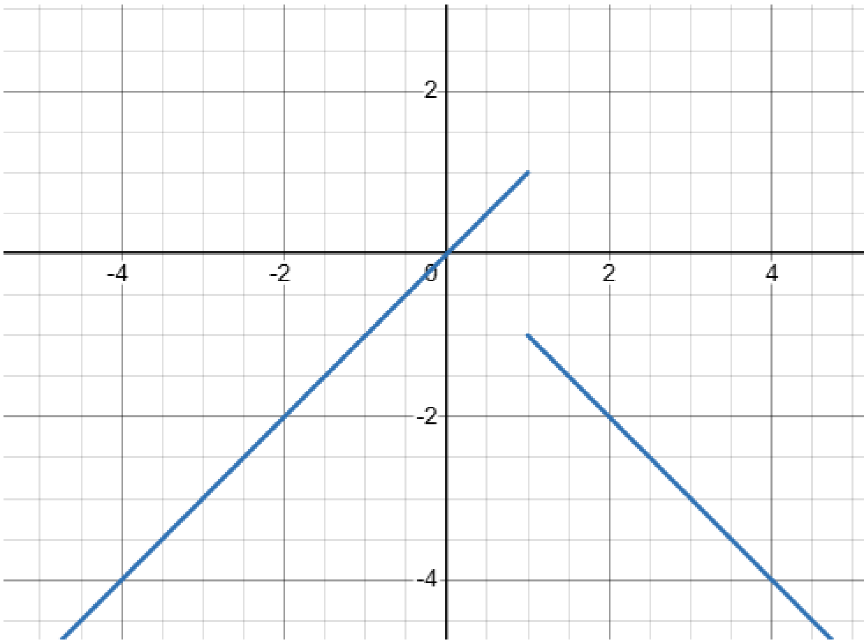
Let's do an example, $f(x)$. x is an input. For x values smaller than 1, the output should be x, and for x values greater than 1, I want $-x$ to be the output.
Deal with the intervals first. These are $x < 1$ and $x > 1$, respectively.
The final step is to write it down in correct notation, with equations next to their interval.

$$f(x) = \begin{cases} x & x < 1 \\ -x & x > 1 \end{cases}$$

$$If\ x < 1,\ \text{the output is just } x.$$

$$x > 1,\ \text{the output is} - x$$

$x = 1$, no output. It's undefined. 1 isn't less, or greater than 1.



Let's reel it back in. We want to calculate the point the animals are at, at t seconds.
Remember the two phases - the **acceleration phase**, and the **constant speed phase**.

In the **acceleration phase**, the distance covered is $s = 0.5at^2\left(s = 0.5\ \times\ a\ \times\ t^2\right).$ We can calculate this phase's duration, as we have the top speed (v) and acceleration (a). $t_{acceleration} = \frac{v}{a}$. So, for any time less than this, we use $0.5at^2$ for the distance, as acceleration is happening!

However, in the **constant speed phase**, we're dealing with a different beast. The distance during acceleration needs to carry over to the second phase. To nobody's surprise, the journey doesn't restart when the timer hits the special number.

Assume the animals have run for t seconds, at their full speed v. So, the distance travelled is tv. No, not the telly.
But, for the first $\frac{v}{a}$ seconds (acceleration phase), they accelerate, and cover less ground than if they were at full speed throughout. The shortfall is $0.5at^2$, the distance under acceleration, so after taking it away, we get the total distance travelled after t seconds.

Now, construct the piecewise function s.

Our intervals are the acceleration, and constant speed phase.

$$t_{acceleration} = \frac{v}{a},\ \text{the time needed to hit top speed.}$$

We're in the acceleration phase, if $\ t \leq t_{acceleration}$.

We're in the constant speed phase, if $t \geq t_{acceleration} -$ the acceleration time has ended.

Now, match our earlier formulas to each interval.

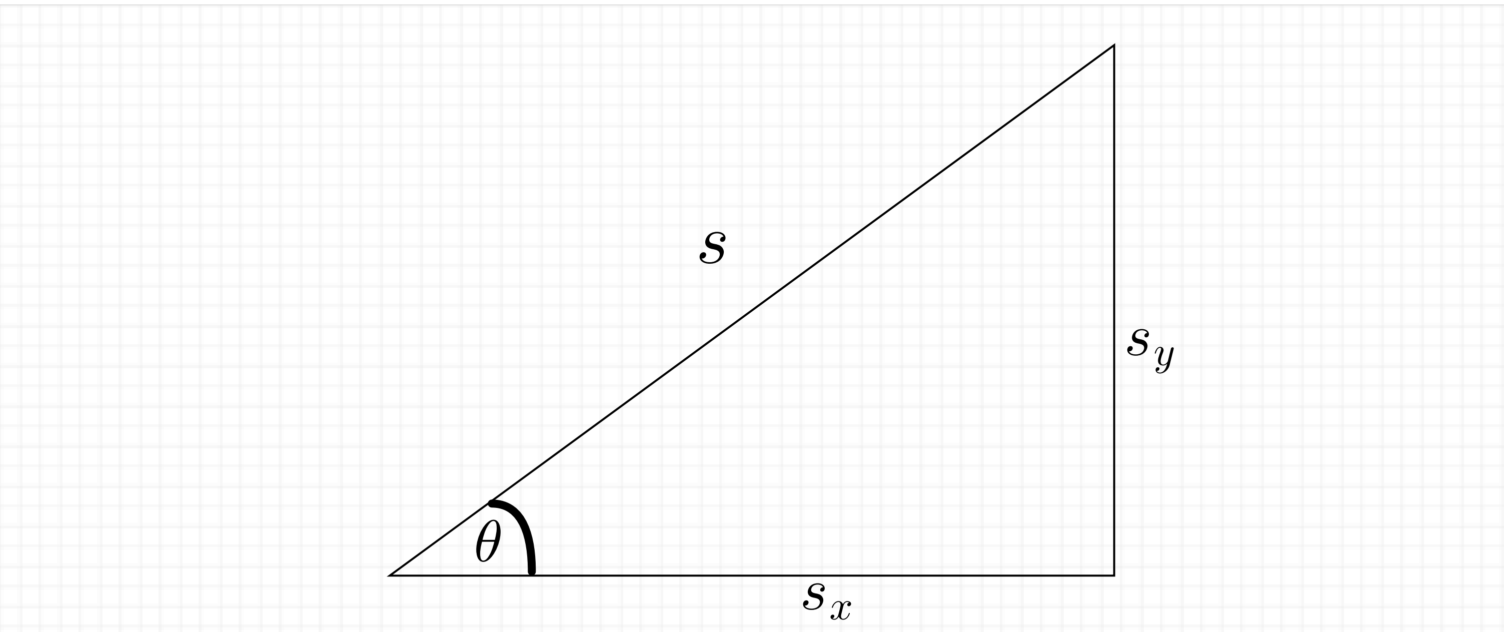
$$s(t) = \begin{cases} 0.5at^2 & t \leq t_{acceleration} \\ tv - \dfrac{v^2}{2a} & t \geq t_{acceleration} \end{cases}, \; where \; t_{acceleration} = \frac{v}{a}$$

Substitute our values in, and make functions $C \; and \; G$ (Cheetah, Gazelle).

$$C(t) = \begin{cases} 3.75t^2 & t \leq 3.6 \\ 27t - \dfrac{729}{15} & t \geq 3.6 \end{cases},$$

$$G(t) = \begin{cases} 2.5t^2 & t \leq 4.4 \\ 22t - \dfrac{484}{10} & t \geq 4.4 \end{cases}$$

Returning to what an 'impact' is, the animals' x, and y co-ordinates must match, at a given time.
There are several impact scenarios.
A, both are accelerating.
B, one is at top speed, and the other is accelerating.
C, both are at top speed.
Before considering them, break the generic distance s the cheetah travels at an angle $\theta°$, into the x and y components. We're doing this, so we can later match the x and y co-ordinates of both animals to ensure they arrive at the same point.



s_x represents the horizontal distance, and s_y represents the vertical.

s_x

s_x is the adjacent side, and we have the hypotenuse. This requires the $\cos$ function.

$$\cos(\theta) = \frac{adjacent}{hypotenuse}$$

$$\cos(\theta) = \frac{s_x}{s}$$

Now, multiply both sides by s to isolate s_x

$$s \times \cos(\theta) = s_x$$

s_y

s_y involves the hypotenuse, and the opposite side. $\sin$ it is!

$$\sin(\theta) = \frac{opposite}{hypotenuse}$$

$$\sin(\theta) = \frac{s_y}{s}$$

Multiply both sides by s to isolate s_y

$$s \times \sin(\theta) = s_y$$

Now, we can dig into the scenarios.

A

Is this even possible...?

Imagine our gazelle's being held down by Thor. It can't move. Cheetah? More like cheater.

The distance the cheetah travels while accelerating is $s = 0.5at^2$, or $3.75 \times t^2$ with $a = 7.5$ across 3.6s.

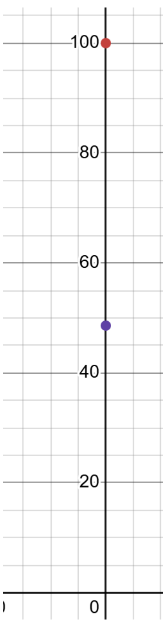
$$\frac{speed}{acceleration} = \frac{27}{7.5} = 3.6.$$

So, now knowing t, alongside a, $s = 0.5 \times 7.5 \times (3.6)^2$

$0.5 \times 7.5 \times (3.6)^2 = 48.6$m.

Not even half of 100m.

It can't reach a frozen gazelle while accelerating. No chance if it's moving across.



B

Nope, useless. While accelerating, the cheetah covers only 48.6m. This scenario would only be remotely possible if the gazelle headed towards the cheetah, but our configuration involves just horizontal movement.

$\textcircled{C}$

A plausible scenario - both at full speed.
So, disregard the first interval of our functions. We only want the second interval, which covers travel at full speed.
Now, return to our graphical model:

- The gazelle runs horizontally. Only the x co-ordinate changes. So, the co-ordinate at a given time is $(G(t),\ 100)$.
- The cheetah runs at an angle $\theta°$. Both the x and y co-ordinates change. Let's use the co-ordinate dissection above, to get $(C(t)\ \times \cos(\theta),\ C(t)\ \times \sin(\theta))$ as the co-ordinate, after t.

Let's replace the functions with the second equations from the piecewise functions, as this scenario is the second phase.
Gazelle: $(22t\ -\ 48.4,\ 100)$
Cheetah: $(\ \cos(\theta)\ \times\ (27t - \ 48.6),\ \sin(\theta)\ \times\ (27t - \ 48.6))$

Final few steps for the meal.
Equate the x and y co-ordinates from the two setups, and isolate the trigonometric functions.

$$\textcircled{x}$$

$$22t\ -\ 48.4\ =\ \cos(\theta)\ \times\ (27t - 48.6)$$

Divide both sides by $27t - 48.6$ to isolate $\cos(\theta)$

$$\frac{22t - 48.4}{27t - 48.6} = \cos(\theta)$$

$$\textcircled{y}$$

$$100 = \sin(\theta)\ \times\ (27t - 48.6)$$

Divide both sides by $27t - 48.6$ again

$$\frac{100}{27t - 48.6} = \sin(\theta)$$

We got there.

$$\frac{22t - 48.4}{27t - 48.6} = \cos(\theta)$$
$$\frac{100}{27t - 48.6} = \sin(\theta)$$

Let's exploit the infamous $\sin^2(\theta) + \cos^2(\theta) = 1$!
Square the expressions equal to $\cos(\theta)$ and $\sin(\theta)$, and add them together. This should equal 1.
Remember that $\sin^2(\theta)$ and $\cos^2(\theta)$ are equal to $(\sin(\theta))^2$ and $(\cos(\theta))^2$ respectively.

$$\frac{22t-48.4}{27t-48.6}=\cos(\theta)$$
$$\frac{100}{27t-48.6}=\sin(\theta)$$

$$\sin^2(\theta)+\cos^2(\theta)=1$$
Sub in the equations for $\sin(\theta)$ and $\cos(\theta)$.

$$\left(\frac{22t-48.4}{27t-48.6}\right)^2+\left(\frac{100}{27t-48.6}\right)^2=1$$

I care about my eyes, but also yours. So, let's do a substitution.
We can let a variable, u, equal the denominator $27t-48.6$.

$$\left(\frac{22t-48.4}{27t-48.6}\right)^2+\left(\frac{100}{27t-48.6}\right)^2=1$$
$$u=27t-48.6$$
Replace with u now!

$$\left(\frac{22t-48.4}{u}\right)^2+\left(\frac{100}{u}\right)^2=1$$

$$\frac{(22t-48.4)^2}{u^2}+\frac{10000}{u^2}=1$$
The denominators are the same, combine the fractions.
$$\frac{(22t-48.4)^2+10000}{u^2}=1$$
Multiply both sides by u^2

$$(22t-48.4)^2+10000=u^2$$
It's a good time to replace u back with $27t-48.6$

$$(22t-48.4)^2+10000=(27t-48.6)^2$$
Expand both brackets.

$$484t^2-2129.6t+12342.56=729t^2-2624.4t+2361.96$$
Rearrange. Take one side away from the other. We'll take away the right from the left.

$$484t^2-2129.6t+12342.56-\left(729t^2-2624.4t+2361.96\right)=0$$
Simplify like terms, resulting in:
$$-245t^2+494.8t+9980.6=0$$

$-245t^2+494.8t+9980.6=0$
There appears to be a quadratic in the savanna... It's of the form at^2+bt+c.
Never forget one lesson in life.

All roads lead back to the quadratic formula.

We use the formula to solve for values of t, when the quadratic $= 0$.

$$t = \frac{-b \pm \sqrt{b^2 - 4 \times a \times c}}{2 \times a}$$

$$a = -245$$
$$b = 494.8$$
$$c = 9980.6$$

Sub in our values to find t. Pay attention to the $\pm$; we need to find two values.
Calculate it twice, with the $+$ and $-$.

$$t_1 = \frac{-494.8 - \sqrt{(494.8)^2 - 4 \times (-245) \times (9980.6)}}{-490} = 7.471\text{s}$$

$$t_2 = \frac{-494.8 + \sqrt{(494.8)^2 - 4 \times (-245) \times (9980.6)}}{-490} = -5.45\text{s}$$

We'll discard t_2 – you can't have negative time!

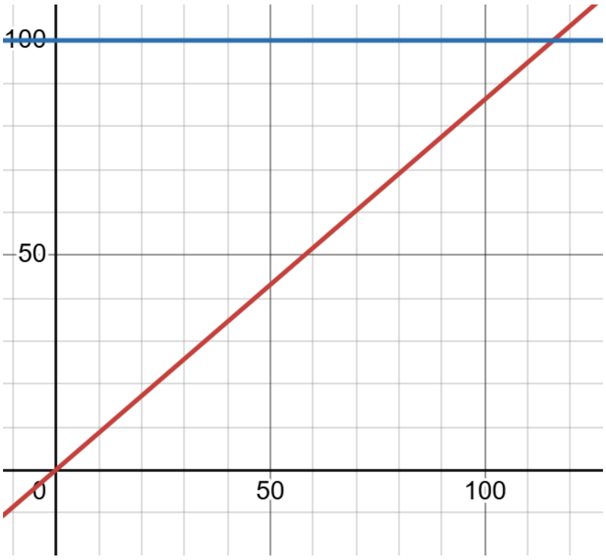
We have t. But not the angle.
Pick out an earlier equation to sub t back into.

$$\frac{100}{27t - 48.6} = \sin(\theta)$$

After substituting t back in, we have an equation to unpack θ.

$$\theta = \sin^{-1}\left(\frac{100}{27 \times (7.471) - 48.6}\right) = 40.78°$$

Lo and behold, the cheetah must travel at $40.78°$ to the horizontal.



Interestingly, this is around $5.35°$ greater than our initial scenario. It makes sense - without a constant $5ms^{-1}$ edge, the cheetah must aim further ahead to compensate.

The simplicity

Perhaps, the simplicity of life is the key observation here.
When a cheetah is sprinting along in the yellow savanna - are there millions of equations firing through its head?
When you're catching some keys thrown your way, are you thinking about Apollonius' circles?
The answer, I hope at least, is a strong "no".

What both of these processes represent is hundreds, if not thousands of centuries of mathematics, executed instinctively, perfectly, and in an instant.
These basal traits did not appear from thin air - nature has programmed us to perform these complex geometrical actions, across millions of years of evolution through trial and error, encoding mathematics into our instincts.

The brain, whether human or feline, is an unparalleled calculator. You probably don't realise it, but the next time you catch a ball thrown your way, take a moment to appreciate the math.
You've been a master mathematician all along.