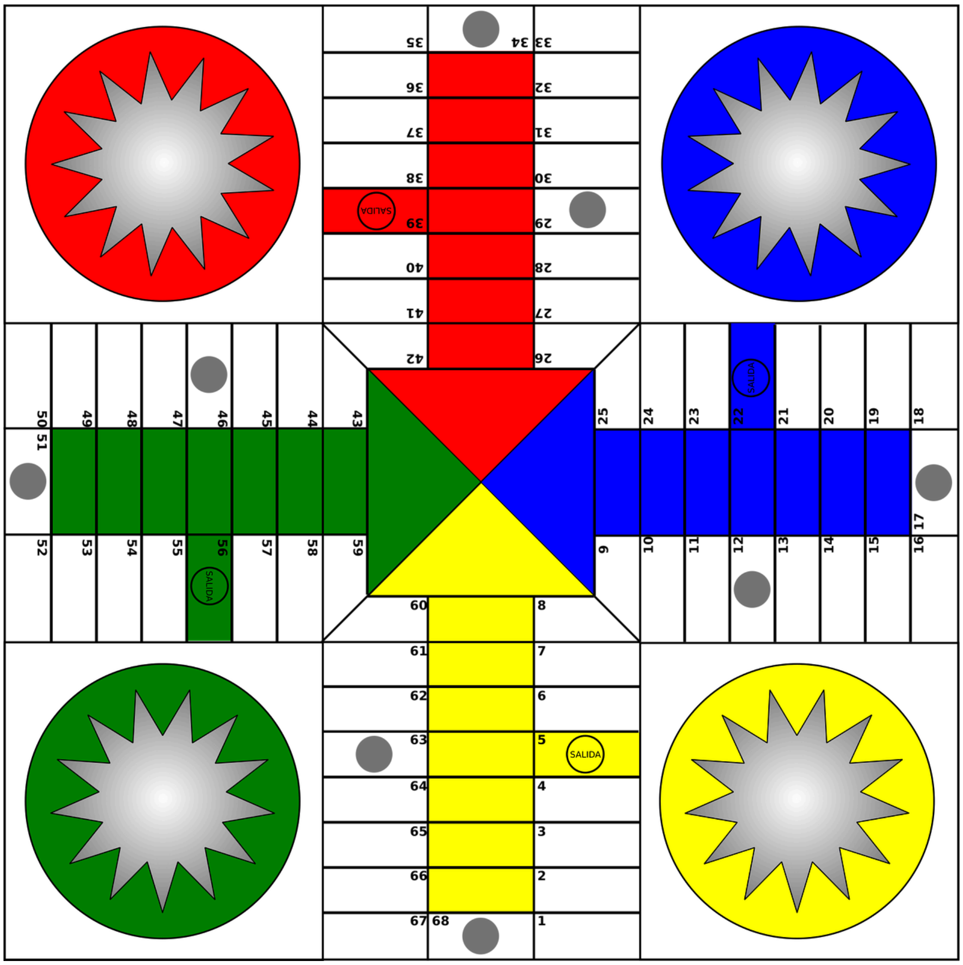
Ludo: A Combinatorial Analysis

By Rian Sarkar



Ludo is a board game which is simple and complex at the same time. It is simple because its rules can easily be understood and lots of it is down to chance; however, it is complex due to the number of unconventional rules. Different rules require different techniques in combinatorics, and the number of considerations to make renders it extremely difficult to predict different aspects of the game such as their expected length, the average number of captures that should occur or the optimal strategy (if there is one). While it is difficult to solve for the entire game, Ludo poses many interesting mathematical problems related to combinatorics and chance. So, let's analyse.

We will follow the ruleset detailed on playiro.com [1] for most rules (but not the three sixes rule - more on that later).

Rolling a Dice

Throughout this essay, it’s assumed that the game is played on a fair, six-sided dice. This means that on any given dice roll, the probability of rolling a 1, 2, 3, 4, 5 or 6 is equal. Since the sum of all probabilities add up to 1, the probability of rolling any number is ⅙. Sorry, but you can’t have a weighted dice up your sleeve.

In Ludo, however, it is possible to get 7. This is because when rolling a 6, you can roll the dice again. Therefore, while the probability of obtaining a 1 is ⅙, the probability of obtaining a 7 is $\frac{1}{6} \times \frac{1}{6} = \frac{1}{36}$ since you must roll a 6 first, and then a 1. Similarly, the probabilities of obtaining 8, 9, 10 or 11 are also $\frac{1}{36}$. It is important to note that you can’t obtain a 6 after a turn since you must roll again after rolling a 6 (unless rolling a 6 pushes one of your pawns home).

Similarly, if rolling two sixes in a row, it is possible to get 13, 14, 15, 16 or 17 in one turn. The probability of getting those scores is ⅙ * ⅙ * ⅙ = 1/216 each since the first two rolls must both be sixes, and then the third roll must be a 1, 2, 3, 4 or 5 respectively. Similar to 6, a 12 cannot be obtained unless pushing a pawn to home.

Just when things were starting to look good, the game-makers decided to introduce a new rule: if you roll three sixes, you get punished. It turns out you can have too much of a good thing!

The Three Sixes Rule

When a player rolls three sixes, various penalties can occur. In some rulesets, the player's turn ends; in other rulesets, the player's turn is ignored and lost. According to playiro.com[1], one of the player's counters gets sent back to base (quite harsh). However, in my family, rolling three sixes means your turn resets and you have to roll again. When calculating probabilities, this leads to a fascinating result!

Since the probability of rolling three sixes is 1/216, the probability of getting a 1 now becomes

$$P = \frac{1}{6} + \frac{1}{216} \times \frac{1}{6} + \frac{1}{216} \times \frac{1}{216} \times \frac{1}{6} + \frac{1}{216} \times \frac{1}{216} \times \frac{1}{216} \times \frac{1}{6} \times ...$$

This is because, you can obtain a 1 from a turn by rolling a 1; or by rolling 3 sixes, your turn resetting, and rolling a 1; or by rolling 6 sixes, your turn resetting twice and rolling a 1, etc…

This results in P being an infinite geometric series which allows us to calculate its value by algebra, or, by using the geometric series formula.

$$216P = \frac{1}{6} \times 216 + \frac{1}{216} \times \frac{1}{6} \times 216 + \frac{1}{216} \times \frac{1}{216} \times \frac{1}{6} \times 216 + ...$$

$$= 36 + \frac{1}{6} + \frac{1}{216} \times \frac{1}{6} + \frac{1}{216} \times \frac{1}{216} \times \frac{1}{6} + ...$$

$$= 36 + P \text{ since } P = \frac{1}{6} + \frac{1}{216} \times \frac{1}{6} + \frac{1}{216} \times \frac{1}{216} \times \frac{1}{6} + ... \text{ therefore}$$

$215P = 36$ meaning that

$$P = \frac{36}{215}$$

This is slightly larger than $\frac{1}{6}$ because there are more ways of rolling a 1. This is a characteristic unique to this three sixes rule (in other rulesets, the probability would just be $\frac{1}{6}$) .

In fact, there are more ways of getting any attainable score, therefore the probability of getting other scores from a turn can also be written as infinite geometric series'. Each geometric series evaluates to $\frac{216}{215}$ times the original probability of getting that score.

Now, we can start to look at different parts of the game of Ludo using this rule. During my calculations, I will only take this into account at the end (to avoid messy numbers), by multiplying (or dividing in some cases) my answer by $\frac{216}{215}$.

Scoring a home run

At the end of a game, when a piece is about to get home, you need to roll exactly the amount needed to get home - any more and your turn doesn't count; any less and you just get closer. This makes it difficult to actually reach home in the first place.

To understand this, consider a piece A that is 1 square away from home. It's so close it can almost "taste" home. The only way it can get home is by getting 1 from a turn. The probability for this is $\frac{1}{6}$ (it *is* $\frac{36}{215}$ but we account for that later) - any other roll would not count and the piece would stay 1 square away from home. Let's calculate the average number of turns expected for A to return home.

Let the expected number of turns it takes for the piece to return home be x. We can thereby derive a recursive formula for x:

$$x = \frac{1}{6} \times 1 + \frac{5}{6} \times (x + 1)$$

Since if a 1 is rolled A gets home in one turn, otherwise A has to wait another turn. This equation can be solved to give:

$$\frac{1}{6}x = 1 \Leftrightarrow x = 6$$

Therefore, on average, it should take 6 turns for A to get home from when A is 1 square from home. Due to the three sixes rule, this is instead slightly less than 6 turns:

$$6 \div (\frac{216}{215}) = \frac{215}{36} \sim 5.972 \text{ turns}$$

If A was two squares from home then we can derive a similar recursive formula to solve for x (the number of turns it takes A to reach home).

$x = \frac{1}{6} \times (6 + 1) + \frac{1}{6} \times (1) + \frac{4}{6} \times (x + 1)$

Since if a 1 is rolled, that's one turn gone and A is one square from home (where it's expected to take 6 more turns to get home). If a 2 is rolled, A gets home in 1 turn. If anything else is rolled, the turn would be ignored (too large) and A would stay put, with the one turn being wasted.

This can be solved to give

$\frac{2}{6}x = \frac{7}{6} + \frac{1}{6} + \frac{4}{6} = \frac{12}{6} \Leftrightarrow x = 6$

Also 6 (or accounting for the three sixes rule, 5.972) turns!

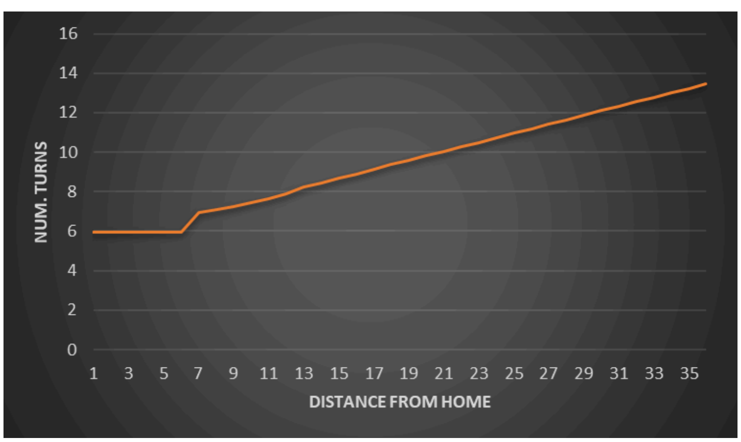
In a similar manner the expected number of turns to get home from other distances can be calculated. For 3, 4, 5 and 6, it is also 5.972 turns - so even if you're 6 squares away, you can "taste home" the same way (since if rolling a 6 gets you home, you don't need to roll again).

This expected number only starts increasing from 7 squares away. By writing a similar-ish equation for x (where x represents the expected number of turns required to get home from 7 squares away), you get:

$x = \frac{5}{6} \times (6 + 1) + \frac{1}{36} \times 1 + \frac{5}{36} \times (x + 1)$, so

$\frac{31}{36}x = \frac{35}{6} + \frac{1}{36} + \frac{5}{36} = 6 \Leftrightarrow x = \frac{216}{31} = 6.968$. Divided by $(\frac{216}{215})$, this yields a total of $\frac{215}{31} \sim 6.935$. With the help of Python, we can extend this process to observe a trend: the further away from home, the longer it should take to get home! Who would have thought?

d	1	2	3	4	5	6	7	8	9	10	11	12
x_d	5.97	5.97	5.97	5.97	5.97	5.97	6.94	7.09	7.25	7.44	7.65	7.89



Cat and Mouse

Let's look at another component in the game of Ludo. When a piece lands on another piece outside a safe zone (a safe zone is a designated part of the board where pieces cannot be captured), that piece is "captured" and sent back to the start. This begs the question: What is the probability a piece gets captured?

Let's visualise a situation with two pieces A and B. B starts 1 square ahead of A. Let's assume there is a safe zone 12 squares away from A. What is the probability that a piece is captured before the safe zone?

To tackle this question, it is important to understand how A and B can end up on the same square.

First, how can A reach a particular square? For example, consider how A can travel 7 squares in 3 turns.

To find the number of ways, you can think about the places A ends up after each of its 3 turns. Using numbers to denote the distance from A's original position, this could be 1,2,7 (if a 1 is rolled, then a 1, then a 5) for example. In fact, this could be any strictly increasing sequence of three positive integers such that the last integer is 7. Therefore, the number of sequences is the number of ways to choose two different unordered integers (the first two) below 7: ${}^{6}C_{2}$. This is $\frac{6\times5}{2}$ (6 possibilities for first number; 5 possibilities for second number; Each pair is repeated twice) = 15.

By these means, it is possible to find the number of ways that A can capture B on any given square. However, calculations get messy when accounting for Ludo's rules such as that the probability A travels 7 in one go is $\frac{1}{36}$, not $\frac{1}{6}$. There can also be repeats (such as if A captures B on the 4^{th} square, A cannot capture B on the 7^{th} square since B has already been captured).

However, there is a simpler, more intuitive way to calculate the probability that A captures B (and also the probability that B captures A). A table could be made, such as the one below, where the row represents A's position and the column B's position (with each entry containing two probabilities - one being the probability that you end up in that position and it's A's turn, the other being the probability you end up in that position and it's B's turn).

To start with, the probability that A is ever in position 0 and B in position 1 with it being A's turn is 1 (since that's where the pieces start). Then from here, A's piece can move after this turn so you can fill in the entries to the right of this entry (where it's B's turn). Similarly, from looking at the probabilities, it is possible to fill in the entries to the right of each entry and below each entry. Since this is an algorithm, I used a computer to automate this (to avoid mistakes and tedious calculations).

place	0	1	2	3	4	5	6	7	8	9	10	11
0	0 0	0 0	0 0.0	0 0.0	0 0.0	0 0.0	0 0.0	0 0.0	0 0.0	0 0.0	0 0.0	0 0.0
1	1 0	0.0 0.1667	0.0 0.1667	0.0 0.1667	0.0 0.1667	0.0 0.1667	0.0 0.0	0.0 0.0278	0.0 0.0278	0.0 0.0278	0.0 0.0278	0.0 0.0278
2	0.0 0	0.0 0.0	0.0278 0.0	0.0278 0.0	0.0278 0.0046	0.0278 0.0093	0.0 0.0139	0.0046 0.0139	0.0046 0.0147	0.0046 0.0108	0.0046 0.0077	0.0046 0.0046
3	0.0 0	0.0 0.0	0.0278 0.0	0.0278 0.0046	0.0285 0.0046	0.0293 0.0094	0.0023 0.0143	0.0069 0.0147	0.0071 0.0112	0.0064 0.0131	0.0059 0.0095	0.0054 0.0063
4	0.0 0	0.0 0.0	0.0278 0.0	0.0278 0.0046	0.0293 0.0093	0.0309 0.0093	0.0047 0.0144	0.0094 0.0152	0.0089 0.0121	0.0086 0.0098	0.0075 0.012	0.0065 0.0081
5	0.0 0	0.0 0.0	0.0278 0.0	0.0285 0.0046	0.0293 0.0094	0.0324 0.0143	0.0071 0.0143	0.0119 0.0155	0.011 0.0128	0.0102 0.0107	0.0095 0.0083	0.0078 0.0107
6	0.0 0	0.0 0.0	0.0278 0.0	0.0293 0.0046	0.0309 0.0095	0.0324 0.0147	0.0095 0.0201	0.0145 0.0201	0.0131 0.0179	0.012 0.0159	0.0109 0.0136	0.0096 0.0109
7	0.0 0	0.0 0.0	0.0 0.0	0.0023 0.0	0.0047 0.0004	0.0071 0.0012	0.0095 0.0024	0.0178 0.0039	0.0114 0.0039	0.01 0.0055	0.0085 0.0064	0.0068 0.0068
8	0.0 0	0.0 0.0	0.0046 0.0	0.0069 0.0008	0.0086 0.0019	0.0104 0.0034	0.0076 0.0051	0.0109 0.0064	0.0129 0.0082	0.0099 0.0062	0.0091 0.0067	0.0079 0.0067
9	0.0 0	0.0 0.0	0.0046 0.0	0.0071 0.0008	0.0083 0.002	0.0096 0.0033	0.0064 0.0049	0.0099 0.006	0.0086 0.0069	0.0113 0.0084	0.0088 0.0059	0.0081 0.006
10	0.0 0	0.0 0.0	0.0046 0.0	0.0064 0.0008	0.0087 0.0018	0.0089 0.0033	0.0052 0.0048	0.0088 0.0057	0.008 0.0063	0.0074 0.0067	0.01 0.0082	0.0079 0.0052
11	0.0 0	0.0 0.0	0.0046 0.0	0.0059 0.0008	0.0075 0.0018	0.0097 0.003	0.004 0.0046	0.0076 0.0053	0.0073 0.0058	0.0071 0.0061	0.0067 0.0062	0.009 0.0076

This table helps identify the probability of captures since the captures all occur when A and B are in the same location.
Therefore, taking the sum of the numbers in the diagonal of this table gives the probability of capture as 0.44097… = 959899898/(6^{12}) and the probability of A capturing B is 0.25231… so quite high indeed!
Repeating this for other values for how far ahead B starts would tell us where the probability A captures B is maximised.

It turns out that if your opponent is 3 spaces ahead of you, you have about a 37% chance of capturing them while they only have a 1% chance of capturing you: ideal for you. The further away from the Goldilocks number 3, the less likely you are to capture them. If you find it is your turn and you and your opponent are both in the safe zone, you might want to move another piece if you can!

Conclusion

All great things must come to an end and unfortunately this essay is no exception. In this essay, we have analysed multiple factors of a game of Ludo, using techniques such as geometric series', recursive relationships and constructing tables. It has been extremely fun, and although you emerge as likely to lose a game of Ludo as you were before, you now know facts to taunt your opponent with.

References

[1] = Ludo Rules → How to Play the Classic Board Game (24/02/26)
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