Fractals: The Shapes In-Between Dimensions

By: Marius Hizon

1. Introduction

Have you ever watched those videos on YouTube where the camera just zooms in on weird-looking shapes and after a few seconds you suddenly see the same shape again? You probably thought you were on drugs or something while watching that (don't worry, even I felt the same at first). Those weird-looking shapes, believe it or not, are actually made from VERY simple rules. I'm talking making a shape (or even just a line) and copy-paste-rotate-attach, or just shape/line-copy-paste-attach, SUPER simple! Or if you're more of a functions guy/gal, something as simple as taking a number, squaring it, adding some fixed number, substituting, and repeating it many times can also give you those weird shapes. This is a relatively new type of geometry called Fractal Geometry, discovered/formed around the mid 1970's by mathematician Benoit **B**. Mandelbrot (take note of the bold B). In this essay, I'll do my best to explain what fractals are, some famous examples, and lastly, how you, yes, YOU can make them yourself!

2. What Is a Fractal?

When you google the definition of a fractal, you'll probably get something like: "A fractal is a shape with self-similar patterns as you zoom in." This is the most common definition. Though this definition isn't always the case, and I'll also show why in the coming parts.

2.1 Fractal Dimension

Let's start by looking at a unit square,

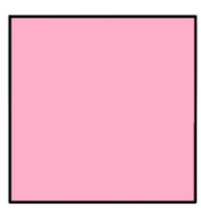


Figure 1: A unit square

We have 1 square with sides equal to 1 and an area equal to 1. Now let's cut it so that it becomes a 3x3 grid

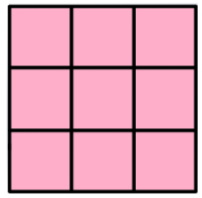


Figure 2: A cut unit square

In this case, we have 9 smaller squares where each side is equal to 1/3 and an area equal to 1/9. Let's now see what happens if we take away the center square, we get

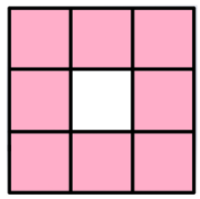


Figure 3: A cut unit square with a center hole

Now here is where it gets interesting. The lengths of the small squares are still the same as figure 2, but to create the original shape, we need 8 copies and to get an area of 1 again. So, the area of the smaller squares must be 1/8.

We know that the area A of a square is L^2 where L is the side length. Figure 1 is easy since L is 1, we get an area A=1. Now let's look at figures 2 and 3 in more depth. So, figure 2 has an area

$$A = L^2$$

$$\frac{1}{9} = \left(\frac{1}{3}\right)^2$$

Which makes sense. Now if we were to use this same formula for figure 3

$$A = L^2$$

$$\frac{1}{8} = \left(\frac{1}{3}\right)^2$$

it does NOT make sense at all! Well, there's actually a different way to interpret the exponent. Let's see what happens if we call it D. This is because we need to find the exponent that let's 1/3 raised to that exponent equal 1/8. We can use some simple algebra to find out the value of D,

$$A = L^D$$

taking the logarithm

$$log(A) = Dlog(L)$$

$$\mathrm{D} = \frac{\log(A)}{\log(L)}$$

and plugging in our values, we get

$$\mathrm{D} = \frac{\log(\frac{1}{8})}{\log(\frac{1}{3})}$$

$$\mathrm{D} = 1.89278926071$$

$$\mathbf{D \approx 1.8928}$$

This D is called the Hausdorff (Fractal) Dimension which means the dimension is apparently between 1 and 2. If you do the same process many times, you eventually get

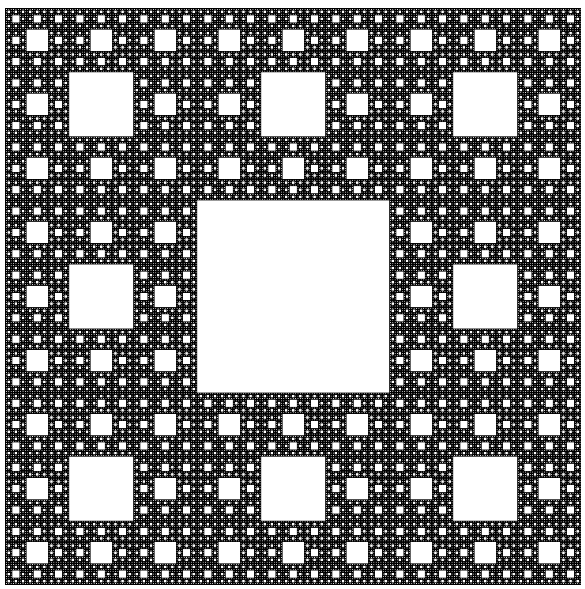


Figure 4: A Sierpinski Carpet

which is called a Sierpinski Carpet and if you look at it, it seems holey-er the smaller you go. The image looks 2D, but its holes seem to make parts of the square vanish, making it look less like a square but more than a line. That's what it means to have a dimension between 1 and 2.

In his book "The Fractal Geometry of Nature," Mandelbrot defined fractals as having a Hausdorff dimension greater than its Topological dimension (that's two definitions). The latter is just a fancy way of saying a whole-number dimension like our familiar 1D, 2D and 3D.

Another way to find D is just make a grid on the shape you're analyzing and count how many boxes that shape is in. Then scale the grid down (or up) by some factor and do it again. The ratio of the larger value to the smaller value happens to be proportional to the scaling factor raised to

D. This is the Box-Counting Method which is kind of an alternate version of the previous method. This is also used mainly for non-self-similar shapes, which I'll be covering next.

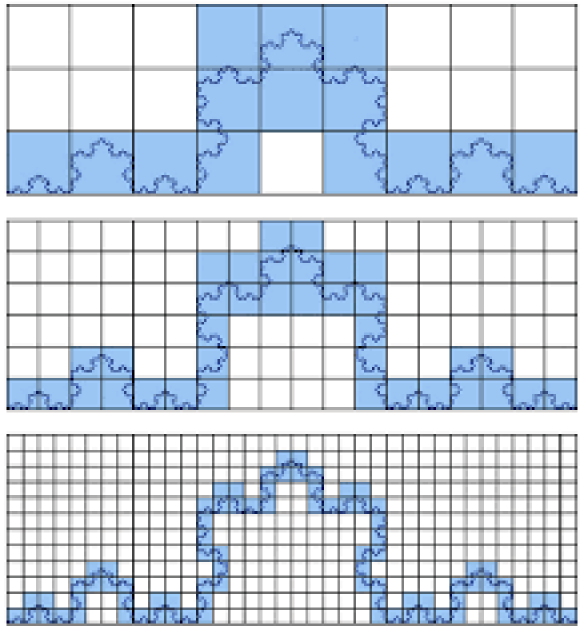


Figure 5: The Box Counting Method

2.2. It's Not Always Perfect

Now this might come as a shocker, but most fractals aren't fully self-similar. When we hear self-similar, we immediately think of seeing the EXACT SAME shape, but scaled down as we zoom in. Though it isn't always like this. Take the natural world for example, especially your own body. Your own blood vessels and arteries are fractals because they branch out into smaller parts (arterioles). While these branching patterns look similar when zoomed in, you'll see that each branch goes in completely different angles. Outside your body, a river is also a fractal, it branches into smaller sections, its smaller branches appear similar, but NOT PERFECTLY. This is what makes the Box-Counting Method good for these shapes.

A common pattern with these examples is consistent COMPLEXITY at different scales. While mathematically you can have perfectly self-similar shapes, in real life, that isn't the case. So, we have another definition "A fractal is any shape with consistent complexity (which may have perfect or imperfect self-similarity) at different scales," (That's three definitions now!).

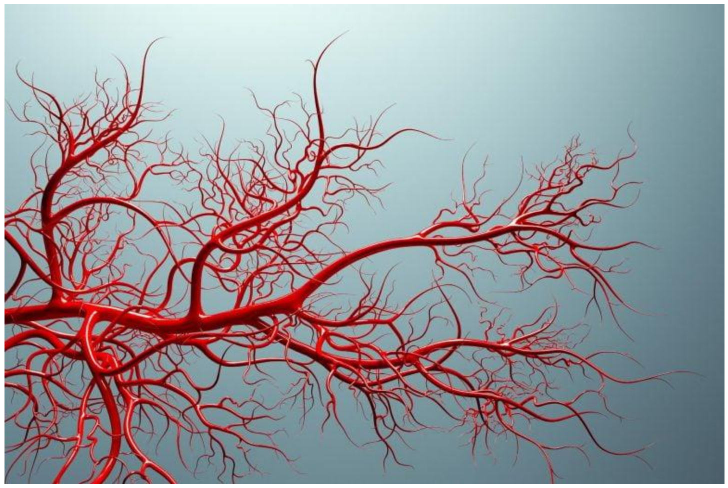


Figure 6: Blood Vessels

Figure 7: A Branching River

2.3. So, What IS a Fractal?

From what we’ve seen so far, fractals have quite a few definitions (and there’s even more if you look it up), but what is the CORRECT one? Well, all of them are correct, the only problem is that each definition is either too restrictive (definition 1) or too vague (definitions 2 and 3). The best way to define it is combining the definitions! So, in this case it would be “Fractals are consistently complex shapes that may or may not be perfectly self-similar and whose Hausdorff Dimension is greater than its Topological Dimension”

3. Notable Fractals

There are many notable fractals (I've already shown the Sierpinski Carpet), but here I'll show you a few (4) more of them.

3.1 The Cantor Set

Take a line with length 1 and cut it into 3 parts, then take the middle part away, then repeat ad infinitum and you get this gnarly thing called the Cantor Set. The dimension is D≈0.6309 which is less than a line but more than a point. And here's a little activity: try to subtract the original line with the sum of the missing segments and see what you get (Hint: It's 3 parts so it's 1/3 then 1/9 but since there's two its 2/9, and you do the rest).

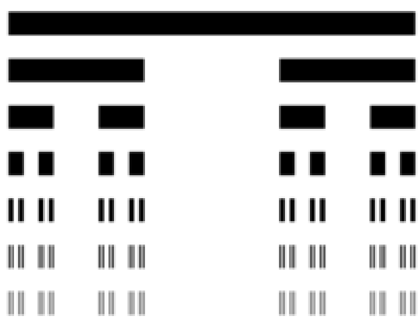


Figure 8: The Cantor Set

3.2 The Sierpinski Triangle

This shape is just "what if we did the same process we did with the Sierpinski Carpet and did it to a triangle?" We get the Sierpinski Triangle which has a dimension of D≈1.585, less than a plane, but more than a line.

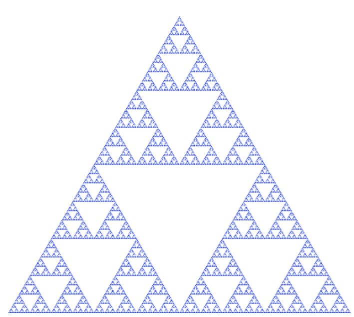


Figure 9: The Sierpinski Triangle

3.3 The Menger Sponge

This here is a 3D version of the Sierpinski Carpet called the Menger Sponge which has D≈2.7268. More than a plane, but less than a solid shape.

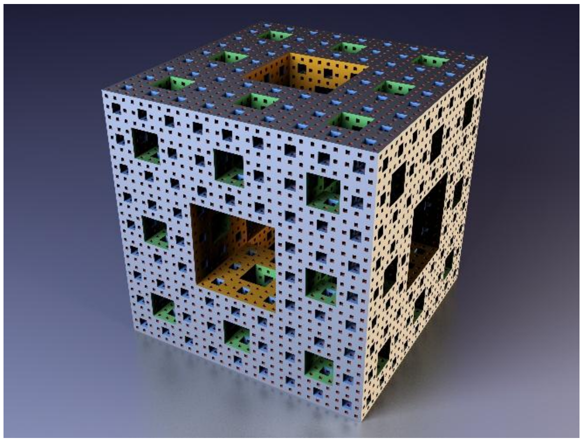


Figure 10: The Menger Sponge

3.4 The Mandelbrot Set

I saved the Golden Boy for last. Named after Benoit himself, the Mandelbrot Set is generated from a specific function which I alluded to in the introduction. If you're curious its dimension is D=2. You can simply think of it as a line that has filled a plane unlike the Sierpinski Carpet and Triangle which doesn't.

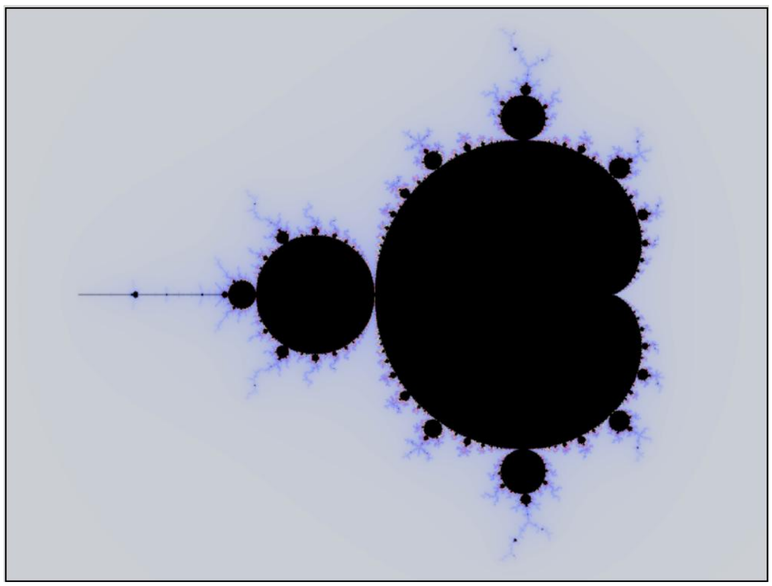


Figure 7: The Mandelbrot Set

The function is this simple equation

$$z_{n+1} = z_n^2 + c$$

z_{n+1} is the value you get after solving the right side, z_n is the previous value of z_{n+1} (z_n is zero if we're just starting), n is the number of iterations, and c is any complex number (a+bi) where i is the square root of -1. The rule of this function is that if the value of c we choose goes to infinity after many iterations, IT IS NOT part of the set, if it oscillates at a certain value, then IT IS part of the set. Let's see two examples showing the two conditions, shall we?

Let's start with a simple value c=1

$$z_2 = z_1^2 + c$$

$$z_2 = 0^2 + 1$$

$$z_2 = 1$$

$$z_3 = 1^2 + 1$$

$$z_2 = 2$$

$$z_4 = 2^2 + 1$$

$$z_4 = 5$$

$$z_5 = 5^2 + 1$$

$$z_5 = 26$$

$$z_6 = 26^2 + 1$$

$$z_6 = 677$$

$$z_7 = 677^2 + 1$$

$$z_7 = 458{,}330$$

Clearly, we see that it will blow up to infinity so 1 IS NOT part of the set. Now let's try another say, c=(0+i)

$$z_2 = z_1^2 + c$$

$$z_2 = 0^2 + i$$

$$z_2 = i$$

$$z_3 = (i)^2 + i$$

$$z_3 = -1 + i$$

$$z_4 = (-1 + i)^2 + i$$

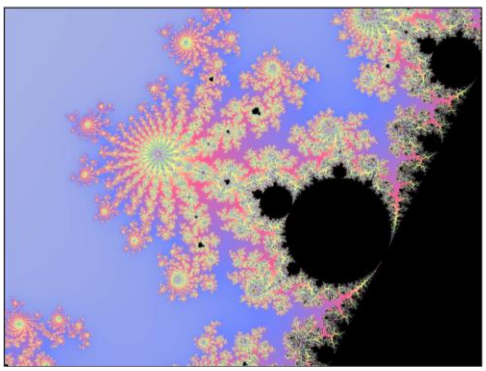
$$z_4 = -i$$

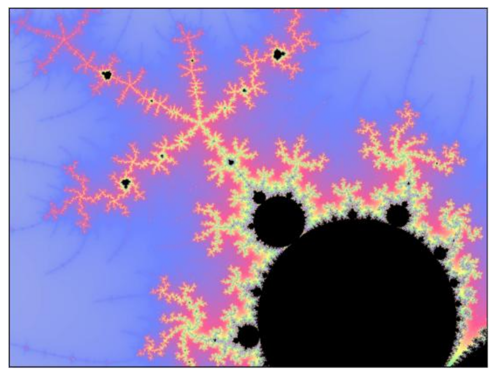
$$z_5 = (-i)^2 + i$$

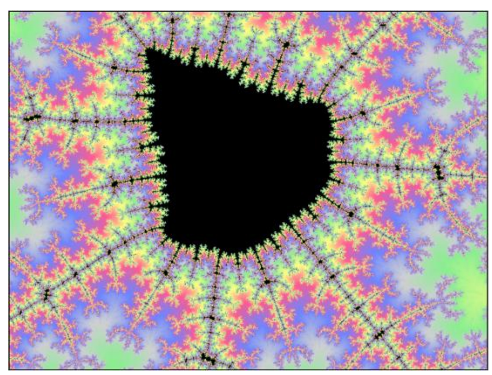
$$z_5 = -1 + i$$

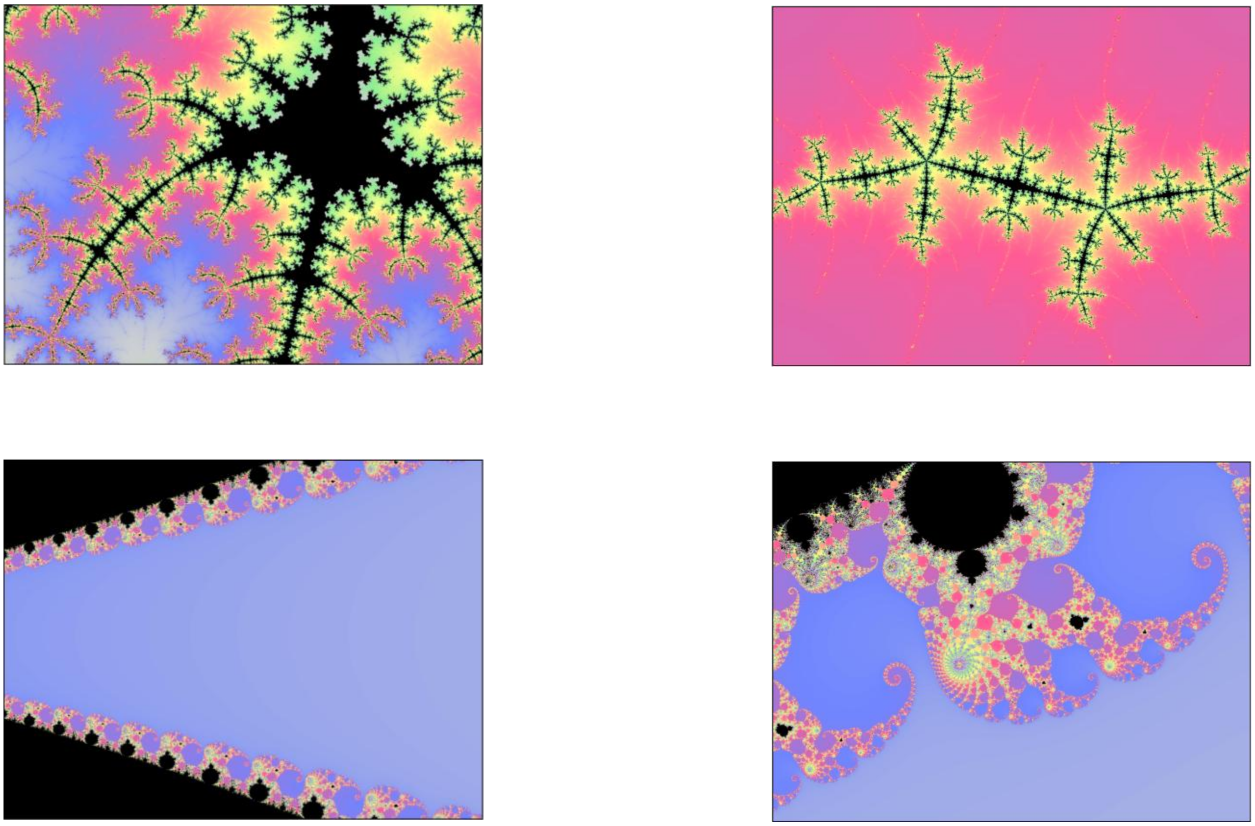
In this case, z oscillates between -i and -1+i, so c=(0+i) IS part of the set.

And since this is the Golden Boy, it's appropriate to see a few zoomed in parts. So here you go!









Figures 8: *Magnificent*

4. Making Your Own Fractals

I know you've been waiting for this, so I'll cut to the chase. You can basically use any software that lets you draw and manipulate the drawings, so things like Microsoft (MS) Paint, Photoshop, even your note-taking app if you can draw shapes and manipulate them. While programming can be used and is honestly much cooler to see, I know not everyone can code, so I'll be using MS Paint since everyone has at least use it before. Anyway, let's get to it!

Let's start by making the Sierpinski Carpet. I'll start with the starting shape which I'll call the root and show a few steps. So here we go!

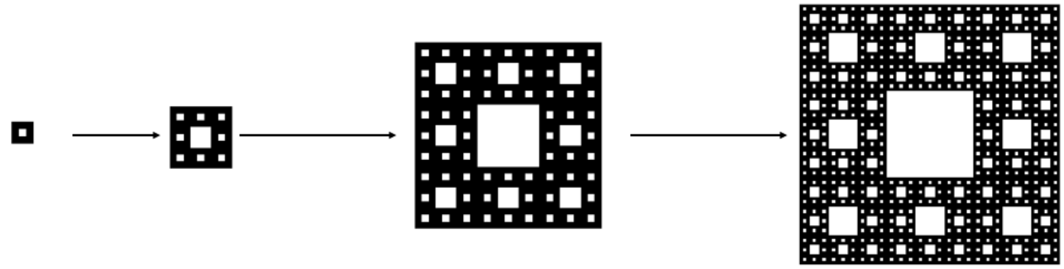


Figure 9a: Steps for the Sierpinski Carpet

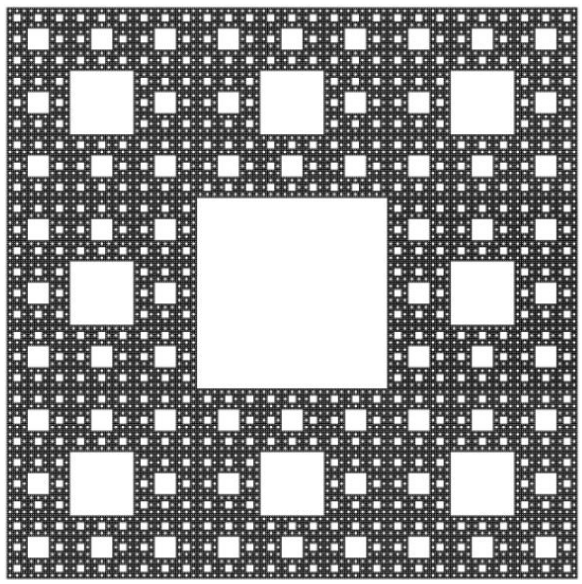


Figure 9b: MS Paint Sierpinski Carpet

Now let's make the Sierpinski Triangle!

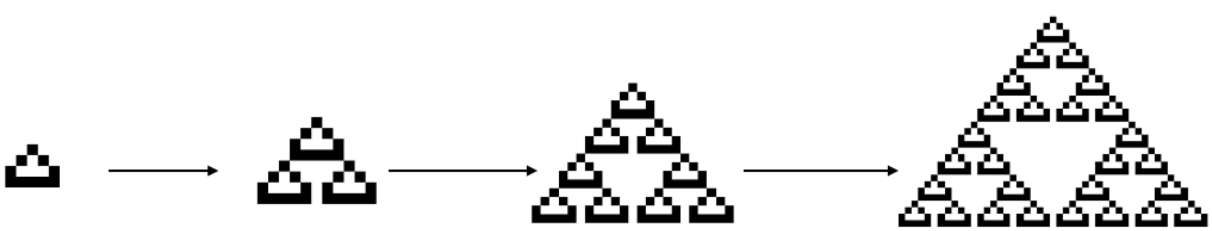


Figure 10a: Steps for the Sierpinski Triangle

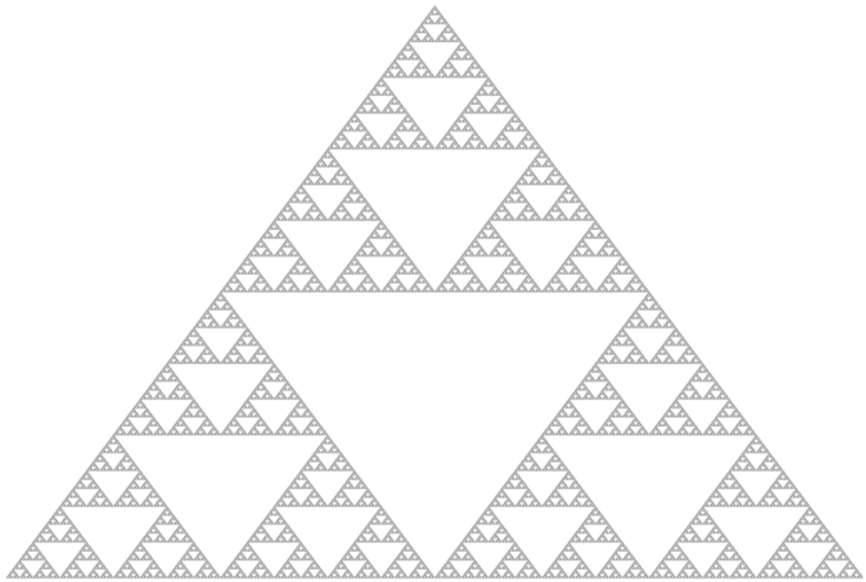


Figure 10b: MS Paint Sierpinski Triangle

This next fractal is called the Dragon Curve. A reminder is always rotate 90 degrees clockwise before connecting the parts.

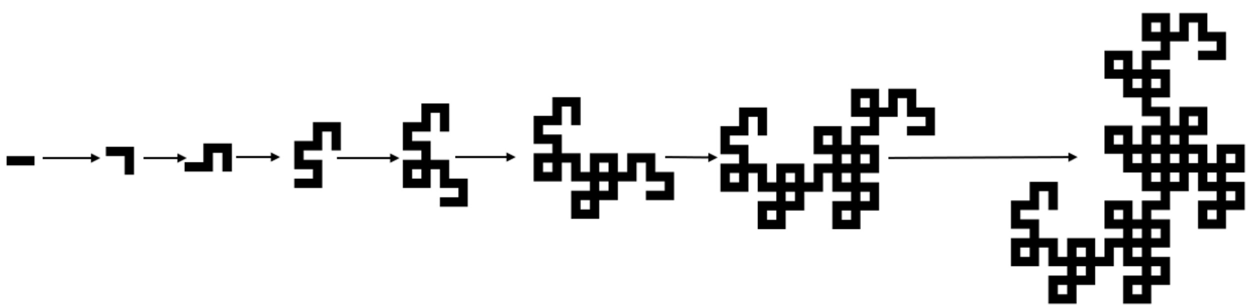


Figure 11a: Steps for the Dragon Curve

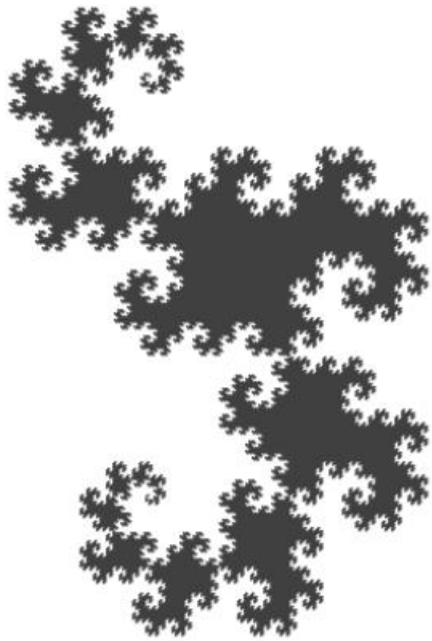


Figure 11b: MS Paint Dragon Curve

Now I'll show you some of my own fractals I made (3)!

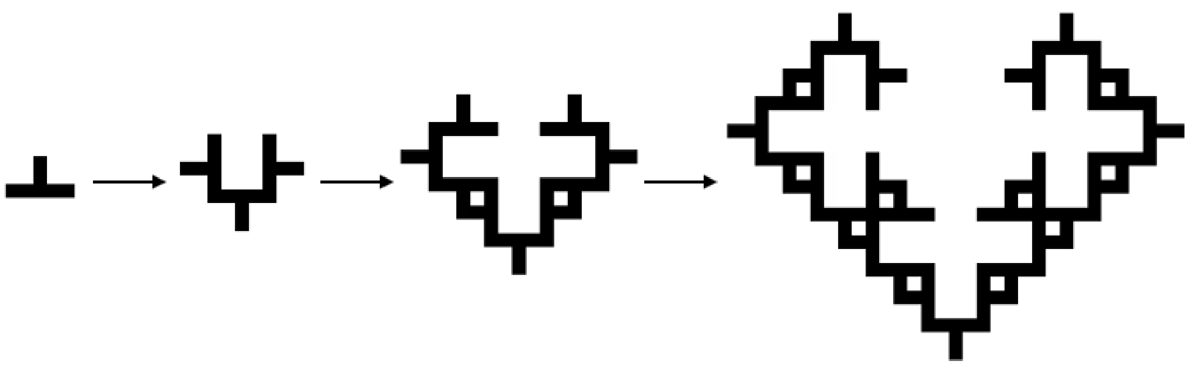


Figure 12a: Steps for the "Manalon Kardia"

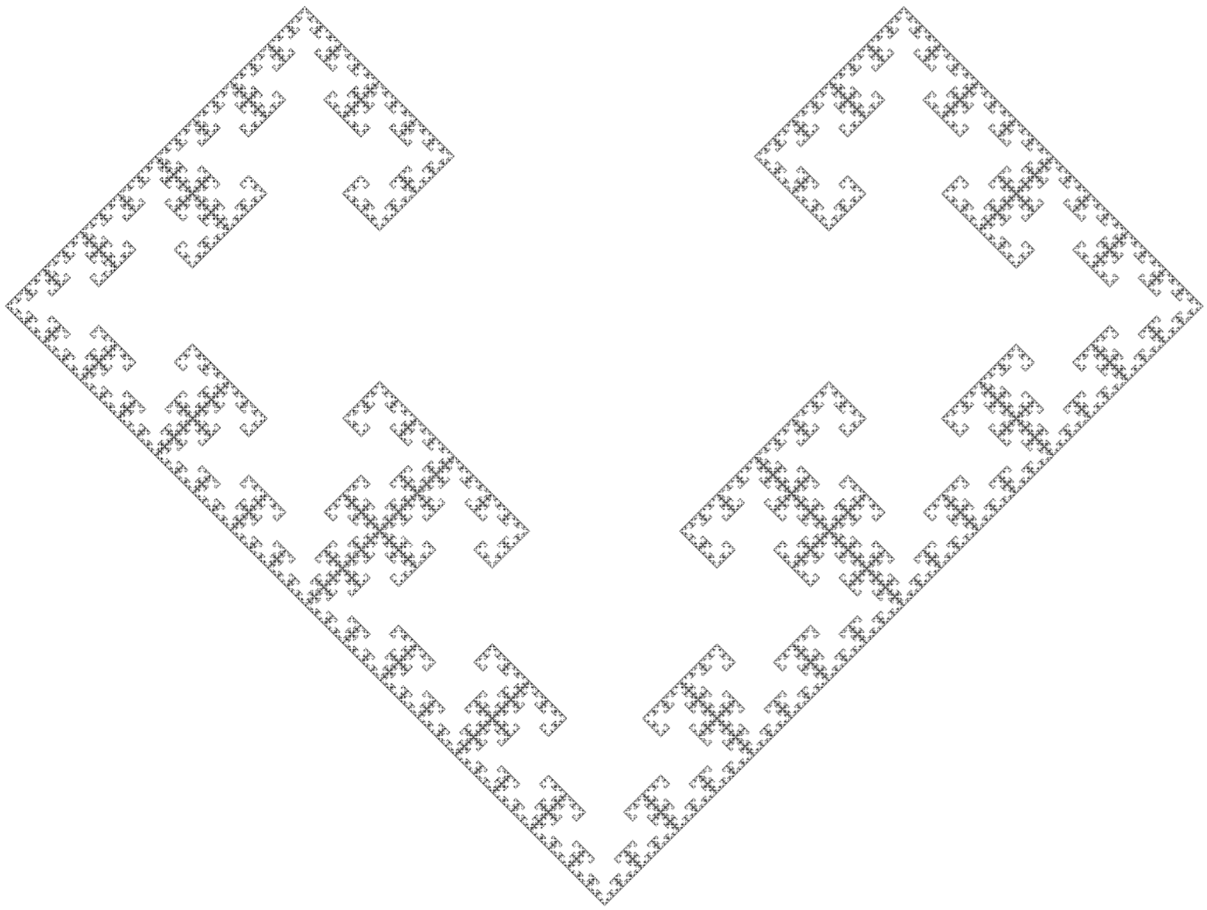


Figure 12b: MS Paint “Manalon Kardia” Fractal

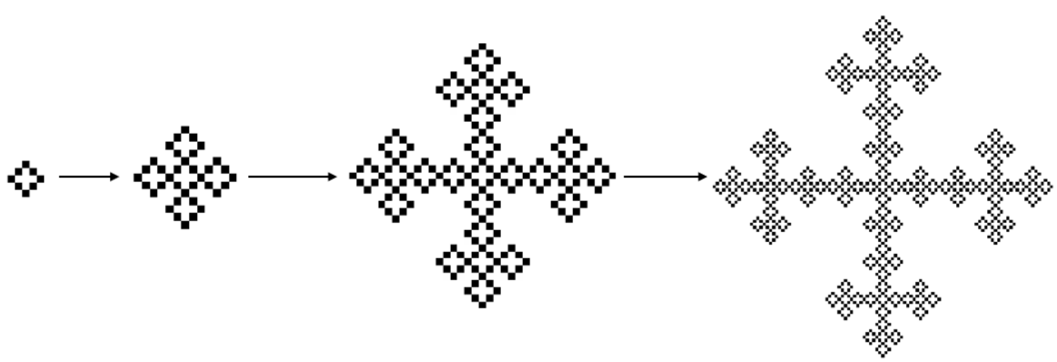


Figure 13a: Steps for the “Laguertan Trèflè”

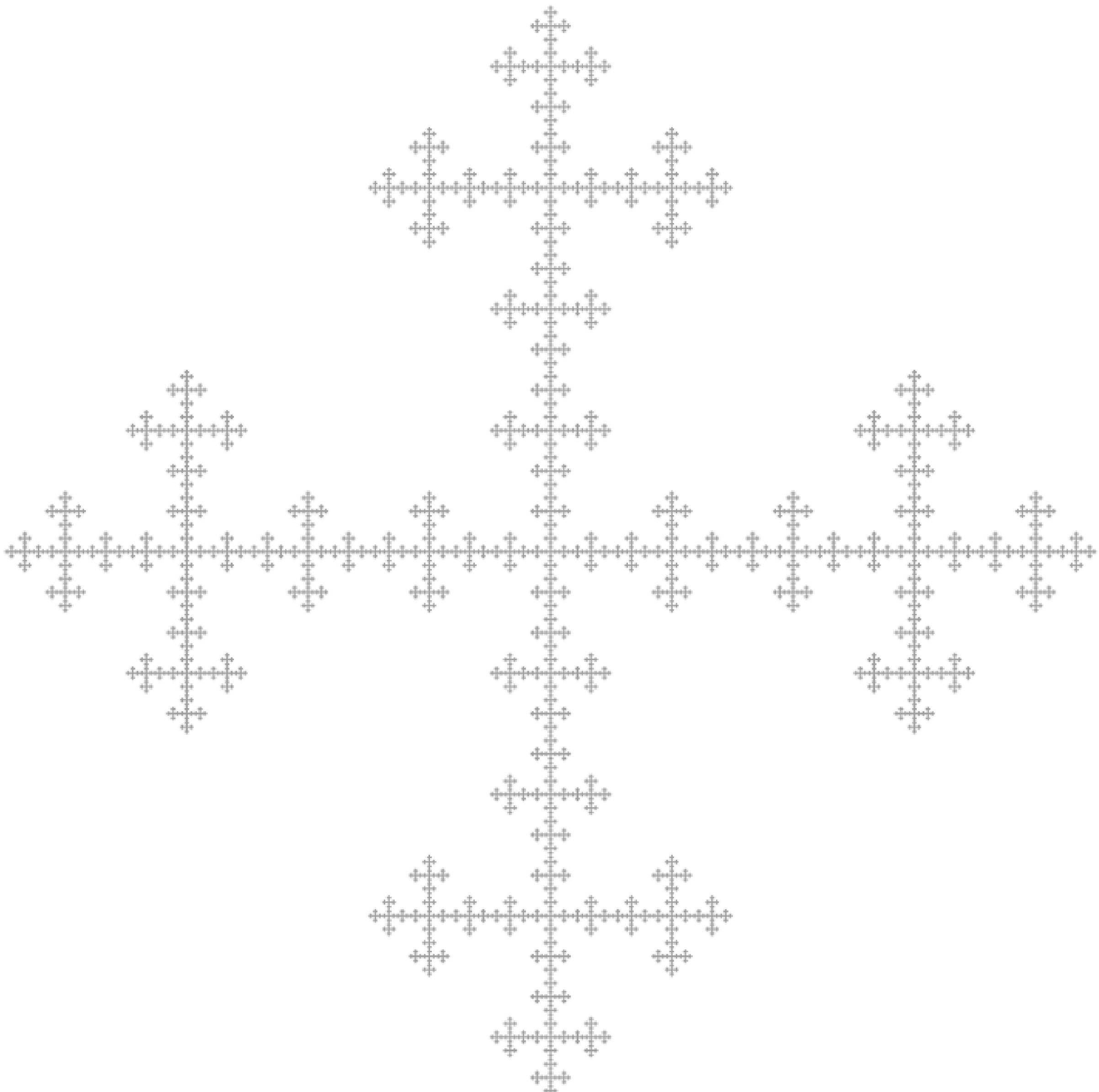


Figure 13b: MS Paint "Laguertan Trèflè" Fractal

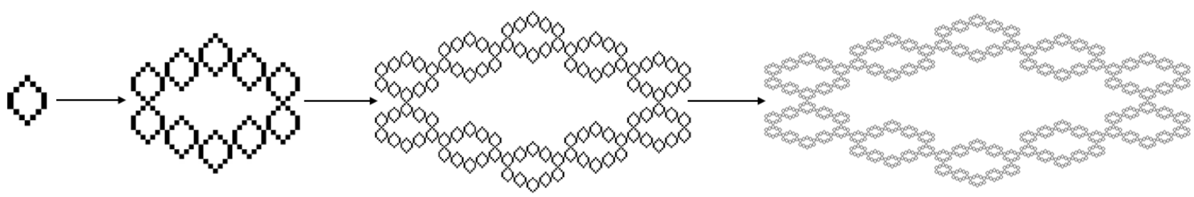


Figure 14a: Steps for the "Aquinian Ring"

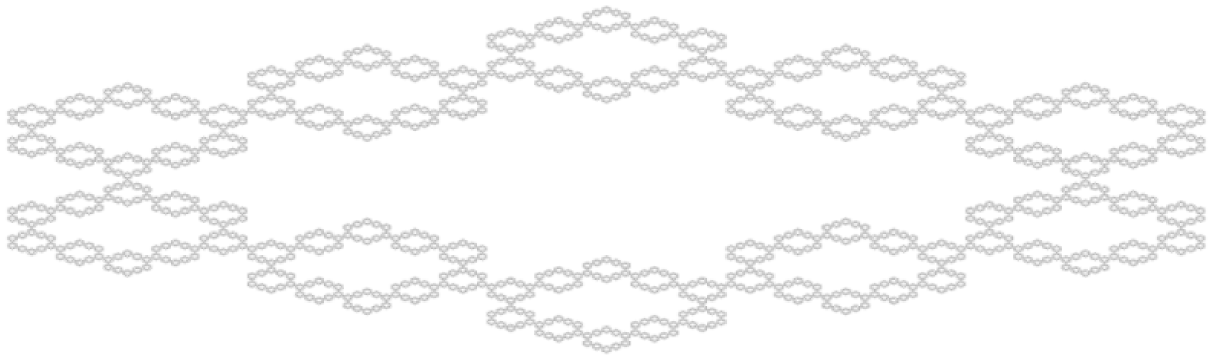


Figure 14b: MS Paint “Aquinian Ring” Fractal

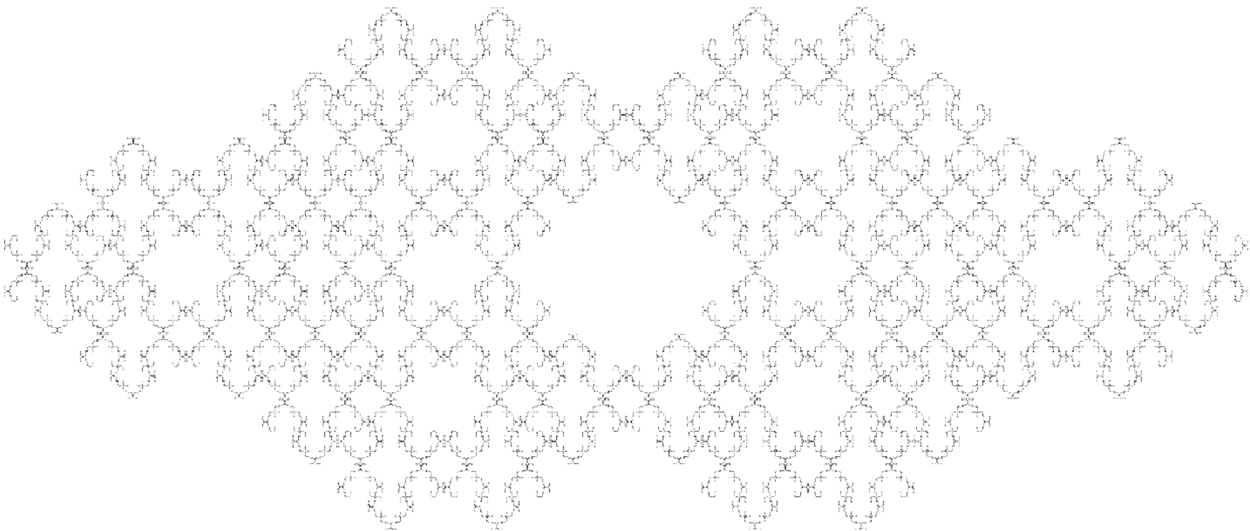
You can honestly do much more than these simpler ones since MS Paint can get wonky with non-90-degree rotations which limits our fractals quite a bit (unless you’re skilled at it). But since MS Paint is in almost every computer and laptop out there, it’s a good start to making your own weird and creative fractals, and later move on to editors with more freedom, or it might even inspire you to learn how to code to make those mind-boggling zooms yourself. On a side note, every time I make lots of iterations, my laptop just lags like crazy, but it’s worth the RAM usage so I don’t mind.

5. Conclusion

The end of the road dear reader and what a journey it has been. I just want to share that after learning about fractals a little deeper, something started to form. If fractals were alive, maybe they are objects of higher or lower dimensions trying to be something they are not (a being of lower or higher dimension). Maybe fractals are the transition from one dimension to the next/previous one, but maybe it’s probably impossible to “ascend” like that, so we get shapes that have infinite complexity and have the strangest properties. I wonder if humans were to try to “ascend” to the fourth dimension, would we look like those fractals? Maybe or maybe not, it’s all theoretical anyway. But I’m not here to be philosophical or make theories, just share how cool and interesting fractals can be.

I hope that this essay gave you at least a tiny speck of curiosity towards these complex shapes and gained new knowledge that dimensions aren’t always whole. Oh! And remember that **B** I told you about, do you know what it means…it’s Benoit B. Mandelbrot! Oh, not laughing? Oh well (fun fact: the B was added by him later and doesn’t mean anything at all). Anyway, thank you so much for reading this essay till the end dear reader and as a parting gift, I'll show you one last fractal I made. Can you guess the root of this fractal? The hint will be a little brain teaser! Well, that is it for me thank you so much again dear reader!

Marius



Zoom in to see me!

References:

[1] Mandelbrot, B. B. (2015) The Fractal Geometry of Nature. Internet Archive. (1983). Internet Archive. https://archive.org/details/fractalgeometryo00beno/mode/2up

[2] Eck, D. J. (n.d.). Mandelbrot Viewer. https://math.hws.edu/eck/js/mandelbrot/MB.html

[3] CodeParade. (2018, August 27). Drawing Fractals in Under 5 Minutes [Video]. YouTube. https://www.youtube.com/watch?v=sFEYQMrWNHU