Symmetry as Automorphisms :
The Conceptual Root of Galois Theory

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Introduction

Humans are naturally drawn to symmetry, with it quietly surrounding our everyday lives : the spiralling patterns found in pinecones and sunflowers; the carefully crafted logos that attract customers to spend; road signs allowing clarity and recognition no matter the language barrier; even time itself, with days, weeks and seasons unfolding in a steady, cyclical pattern. Symmetry is not something we can escape; it lies in the very foundations of the world we experience.

I, too, love symmetry, believing it is one of the most beautiful yet powerful ideas in mathematics : “a type of invariance : the property that a mathematical object remains unchanged under a set of operations or transformations.” In other words, if you perform a specific operation on the object (reflection, rotation, translation), it looks the same as before, preserving the object’s structure. For example, think of a square. If you rotate it 90, 180, or 270 degrees, it occupies the same space and looks identical to before; this is known as rotational symmetry. Or if you reflect the same square over a line passing through the middle / across its diagonals it too remains unchanged ; reflectional symmetry.

Automorphisms formally embody this idea of symmetry as a structure-preserving mapping from a mathematical object to itself. This more abstract idea allows symmetry to be studied beyond geometric mathematics, as an organising principle.

This principle is the central idea behind Galois Theory, developed by Évariste Galois in the early 1830s, which reframed algebra by linking polynomial roots to group theory. This allowed a shift from the classical method of solving polynomials through radicals to solving them by interpreting the relationship between their roots as symmetries.

Symmetry as Automorphisms

In more abstract terms, an automorphism of a mathematical object is a bijective map from the object to itself that preserves all relevant structure. ‘Bijective’ means a function between two sets that pairs every element in one set to another set, leaving no elements unpaired in either set. A function is only bijective if and only if it has an inverse function, as reversibility is a key sign of a bijection. For example, a multiplication by two defines a simple bijection, from the integers to even numbers (with division by two as the inverse function). The meaning of ‘structure’ depends on the context of the question: in geometry it could mean distances or angles, in algebra operations like addition or division, some may say it is objects with rules.

The collection of all automorphisms of a given object forms a group under composition, known as its automorphism group. The collection outlines all the symmetries for the object, capturing the internal structural symmetries of the object. For example, all symmetries of a square, its rotations and reflections, form a group made from rotations and reflections; algebraically, the automorphisms of the group would capture its internal structural symmetries.

Studying an object's automorphism group rather than the object itself shifts the focus from elements of a structure to the transformations that preserve that structure, revealing invariants that provide a tool for classification. This is a cornerstone of a large proportion of abstract algebra, where entire families of structures can be understood based on their symmetry groups. This idea is shown in Galois theory, where the symmetries of polynomial roots are realised as automorphisms of fields.

The Ancient Problem of Polynomial Equations

The beginning of Galois Theory lies in the historic effort of solving polynomial equations. Formulas for quadratic, cubic and quartic equations have already been discovered, with a long-lasting attempt to find a formula for quintic equations and degrees above five being proved impossible.

Galois' shift in thinking moved away from explicit solutions and towards how the roots are related to one another, recognising that the difficult part of solving a polynomial equation was not computational but instead structural. The relationships of roots remain invariant under certain transformations, suggesting that understanding equations means focusing on their symmetries. This shift in focus from more radical to structural naturally flows towards automorphisms.

Field Extensions and Root Symmetries

To show these ideas, Galois theory uses the notion of a field extension. This starts with a base field, like the rational numbers, **Q**, and joins it with the roots of a polynomial. For example, the equation : $x^2 - 2 = 0$ has the field $\mathbf{Q}(\sqrt{2})$, which contains the roots $\sqrt{2}$ and $-\sqrt{2}$.

This shows that we only allow changes that keep all equations true while keeping the structure intact, 'allowed changes' simply being automorphisms. In the case above, there are only two automorphisms : the identity itself and the map sending $\sqrt{2}$

to - $\sqrt{2}$. Since these transformations preserve all algebraic relations, they are deemed true symmetries of the field.

However, not every permutation of the roots is allowed, only the arrangements that preserve the algebraic structure, as these are the ones that ensure the automorphisms of the polynomial reflect genuine symmetries rather than any rearrangements of the roots without caring for keeping the algebraic relationships true.

The Galois Group

The set of all the automorphisms of a field extension that fix the base field forms a group under composition, called the Galois group, which encodes symmetries of the roots of the polynomial equation used and, as a whole, captures the full symmetry of the polynomial. This group, in simple terms, shows how much freedom you have to rearrange the roots, with a smaller group having fewer ways to rearrange, so it is easier to solve. In comparison, a larger, more complicated group will have many rearrangements, so will be harder or even impossible to solve.

This idea is the foundation of Galois Theory, where the properties of a polynomial are determined by the structure of its Galois group. Studying the group of automorphisms acting on roots rather than the roots directly shows how algebraic problems can turn into more group-theoretic ones, which allow different methods to be applied to them. By looking at it this way, Galois theory provides evidence that symmetry (through automorphisms) as a general principle is needed to understand mathematical structure.

Fundamental Theorem and Structural Correspondence

In Galois theory, there are two components : fields (number systems) and groups (all the allowed rearrangements of the roots). The connection between the two is explained by the Fundamental Theorem of Galois Theory, which shows a one-to-one correspondence between the subgroups of the Galois group and the intermediate fields of the extension. In other words, each subgroup corresponds to a subfield. For example, if you begin with a big number system and a big set of symmetries, if you limit the rearrangements you simultaneously get a smaller number system.

This link allows translations between algebraic structures to questions about symmetry groups, showing that the structure of a field extension is closely reflected in the structure of its automorphism group. It also means that by having an understanding of the group, you have insight into the field too. Using the idea of symmetry as automorphisms, this theorem represents a perfect match between groups and fields.

Solvability and Symmetry

A final example of applying Galois theory is how it explains solving polynomials by radicals. A polynomial equation can be solved by radicals if its Galois group has a certain structure, a structure built from simple groups in a particular way that allows it to be shuffled into a formula.

This is evidence of why quintic polynomial equations have no general formula by radicals, as their Galois groups are too complex so are unable to be rearranged into a formula. Therefore, some equations being impossible to solve is not due to a lack of innovation but due to their complex symmetries, meaning a formula cannot exist. By focusing on the structure of the symmetry group instead of directly solving it, you can determine fundamental properties of the equation, showing the power behind having the automorphism perspective.

Conclusion

To finish, I want to say how I believe Galois theory as a whole changed how mathematicians think about equations. It showed that you can understand a concept just by looking at the changes you can make. Thus, leading to some arguing that the most important thing about a mathematical object is it symmetries, as it allows changes without altering the structure of the object.