Mathematically Proving I'm Always In Tune

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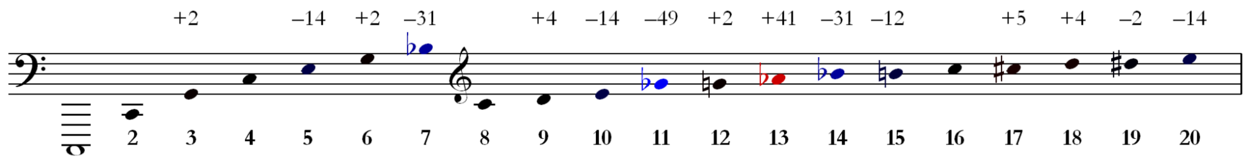


Introduction:

There are various definitions of being "in tune" in music. Being perfectly in tune for even a brief second is impossible due to the constantly changing weather around us. So then, what is classified as "in tune"? Is it within a certain range? To the point where no one can hear a difference? Using the power of math, we can find how we can be "in tune" for any definition of "in tune"!

What do I mean in the title by "*I'm Always In Tune*" after stating it is physically impossible in the last paragraph? That depends on the definition of being "in tune". If the range for being "in tune" is large enough, I can be in tune by playing any note as long as it is within that range. However, what if the range was extremely small, like an error of only 1 cent, or even the exact frequency? We can dive into this problem in a new way, by looking at the harmonic series… of music.

One day, I was studying the concept of the harmonic series of music. The frequencies are the multiples of the fundamental frequency, but it referred to the fact that some of the notes were slightly out of tune. I was initially confused as to why this would happen. How are some harmonic frequencies, such as 1320, a clean multiple of 440, be roughly two cents sharp? My instincts told me to dive deeper into this topic, using **math**.

Now, before the journey starts, here is some musical vocabulary that we will encounter:

- Hz (frequency) - The number of oscillations a sound wave gets per second. The higher the frequency, the higher the note.
- Sharp - when a note has a higher frequency than a reference frequency.
- Flat - when a note has a lower frequency than a reference frequency.
- Harmonic(s) - the series of frequencies that come with a fundamental frequency, which are multiples of the fundamental frequency. Normally, the harmonics are much softer than the fundamental pitch, where the more harmonics you go up, the less audible they become, but they do exist. The fundamental pitch corresponds to the 1st harmonic.
- Octave - the interval of notes that are typically two times in frequency.
- Semitones - the equally spaced notes in an octave (typically 12). Each semitone corresponds to a labeled note: A, A#, B, C, etc.
- Cents - the 100 equally spaced values in one semitone.

Now let's begin.

The Journey Starts:

How might we use math to solve these questions? My thought was to find an equation to find the cent deviations for each of the harmonics. Then, adding constraints to the y-axis to only show the harmonics with the cent deviation less than a certain amount. This way, we can find the exact harmonics where a note might be considered "in tune".

To explain my calculation in an organized manner, I'll show the product, then explain it by going in order.

(Note that we will skip the last two columns for this essay. The reader can attempt to find the equation I used for the last column.)

Harmonic (x)	Frequency $f(x)$	Exact semitones away	in-tune Hz down	in-tune Hz up	cent difference (down)	cent difference (up)	final cent diff	Semitones away	Note
1	440	0	440	440	+0	-0	+0	0	A4
2	880	12	880	880	+0	-0	+0	12	A5
3	1320	19.0195	1318.51	1396.91	+1.9550008	-98.0449991	+1.9550008	19	E5
4	1760	24	1760	1760	+0	-0	+0	24	A6
5	2200	27.8631	2093.00	2217.46	+86.313713	-13.6862861	-13.6862861	28	C#6
6	2640	31.0195	2637.02	2793.82	+1.9550008	-98.0449991	+1.9550008	31	E6
7	3080	33.6882	2959.95	3135.96	+68.825906	-31.1740935	-31.1740935	34	G6
8	3520	36	3520	3520	+0	-0	+0	36	A7
9	3960	38.0391	3951.06	4186.00	+3.9100017	-96.0899982	+3.9100017	38	B7
10	4400	39.8631	4186.00	4434.92	+86.313713	-13.6862861	-13.6862861	40	C#7
11	4840	41.5131	4698.63	4978.03	+51.317942	-48.6820576	-48.6820576	42	D#7
12	5280	43.0195	5274.04	5587.65	+1.9550008	-98.0449991	+1.9550008	43	E7
13	5720	44.4052	5587.65	5919.91	+40.527661	-59.4723382	+40.527661	44	F7
14	6160	45.6882	5919.91	6271.92	+68.825906	-31.1740935	-31.1740935	46	G7
15	6600	46.8826	6271.92	6644.87	+88.268714	-11.7312852	-11.7312852	47	G#7
16	7040	48	7040	7040	+0	-0	+0	48	A8

What is this insanity? Let's go from left to right to understand this.

We start with the harmonic number x in column 1.

$x = \text{ } harmonic\ number$

Now we need the frequencies of each of the harmonics in the next column. In the example table above, the fundamental frequency (f_1) is set at 440, but this can be any frequency, so we'll call it z. The column is then calculated as zx.

$z = fundamental\ frequency$

For the next column, we need to find how far a harmonic note is from the original note. To do this, we have to know the formula for finding how logarithmically far a frequency is from a reference frequency. This is found by the formula:

$$s = 12\, log_2\left(\frac{f}{f_1}\right)$$

where s is the semitones away. What are f and f_1?

f, in this case, represents the frequency from column 2 where it was notated as zx. f_1 is then the tuning frequency, typically set at 440 Hz. However, this can still be changed to a different tuning, such as the rather uncommon 432 Hz. We'll set the tuning frequency as r. Combining all of this, the equation for the column becomes:

$$s = 12\, log_2\left(\frac{zx}{r}\right)$$

$r = tuning\ frequency$

Columns 4 and 5 are basically the reverse of what we did before. Column 4 returns the semitones away to the note below and finds the frequency of that note by using the reverse of the last formula. Column 5 instead takes the note above the semitones away and finds the frequency. Each of them does this with the floor and ceiling functions. Therefore, the equations are:

$$f_{down} = r \cdot 2^{\frac{\left\lfloor 12\, log_2\left(\frac{zx}{r}\right)\right\rfloor}{12}} \qquad f_{up} = r \cdot 2^{\frac{\left\lceil 12\, log_2\left(\frac{zx}{r}\right)\right\rceil}{12}}$$

The next two columns are to find the cent differences of each note, compared to the rounded-down (column 4) and rounded-up (column 5) frequencies. There are 100 equally spaced cents in one semitone, so the formula for finding cent deviation is the same as finding the semitones away, but multiplied by 100:

$$d = 1200\, log_2\left(\frac{f}{f_1}\right)$$

where d represents the cent deviation. This time, what are f and f_1? f is the frequency created by each of the harmonics (zx), while f_1 is the frequency of the rounded-down/rounded-up note. Combining these gives the formulas for columns 6 and 7:

$$d = 1200\, log_2\left(\frac{zx}{r \cdot 2^{\frac{\left\lfloor 12\, log_2\left(\frac{zx}{r}\right)\right\rfloor}{12}}}\right)$$

$$d = 1200\, log_2\left(\frac{zx}{r \cdot 2^{\frac{\left\lceil 12\, log_2\left(\frac{zx}{r}\right)\right\rceil}{12}}}\right)$$

The next column is to see which of the two equations from the last step was more accurate. By taking the absolute values and comparing them, we can see which side was closer to the intended pitch. For example, you would want to say 1.955… cents sharp instead of 98.044… cents flat since 1.955… cents is musically closer to a note than 98.044… cents. Then we can just show the one that was more accurate. We can do this with one piecewise function that looks like this:

$$f(x) = \begin{cases} 1200\, log_2\left(\frac{zx}{r \cdot 2^{\frac{\left\lfloor 12\, log_2\left(\frac{zx}{r}\right)\right\rfloor}{12}}}\right) & \text{if, } \left|1200\, log_2\left(\frac{zx}{r \cdot 2^{\frac{\left\lfloor 12\, log_2\left(\frac{zx}{r}\right)\right\rfloor}{12}}}\right)\right| \leqslant \left|1200\, log_2\left(\frac{zx}{r \cdot 2^{\frac{\left\lceil 12\, log_2\left(\frac{zx}{r}\right)\right\rceil}{12}}}\right)\right| \\ 1200\, log_2\left(\frac{zx}{r \cdot 2^{\frac{\left\lceil 12\, log_2\left(\frac{zx}{r}\right)\right\rceil}{12}}}\right) & \text{if, } \left|1200\, log_2\left(\frac{zx}{r \cdot 2^{\frac{\left\lfloor 12\, log_2\left(\frac{zx}{r}\right)\right\rfloor}{12}}}\right)\right| > \left|1200\, log_2\left(\frac{zx}{r \cdot 2^{\frac{\left\lceil 12\, log_2\left(\frac{zx}{r}\right)\right\rceil}{12}}}\right)\right| \end{cases}$$

This ensures that the value closer to 0 gets shown while the other doesn't. Here's a challenge to the people reading this: try to simplify this monstrosity of an equation. The answer is shown on the next page, so check your answers if you wish.

$$u = 12\, log_2\left(\frac{zx}{r}\right)$$

$$f(x) = \frac{1200}{12}(u - round(u))$$

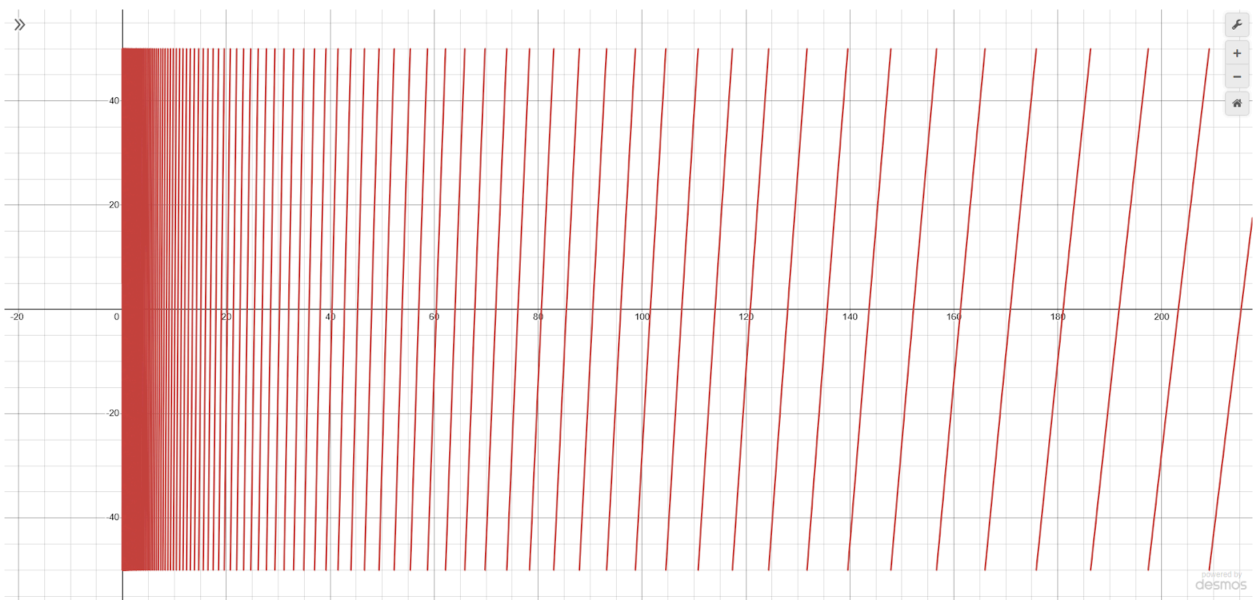
If you were able to get an equivalent of this equation, congratulations, you're absolutely insane.

Referring back to what each one of these variables represents, can you see what "u" as a whole represents now?

u represents the number of semitones away a note was from the tuning frequency! The parentheses part of the formula will either give a positive value (when the harmonic is sharp) or a negative value (when the harmonic is flat). Since these variables are not restricted, we can now say that we have created a generalization for the cent differences for the different harmonics.

So… How Does This Help?:

Now that we understand it, let's graph it for the simplest case, where z is the note 440 and r is also the conventional 440:



This graph uncovers a ton of things about the harmonic series of music. The generalization wasn't one smooth line that connects every value; instead, it was a series of unnoticeably curved lines that each represented a different semitone. This means that you could hit any note in tune if you play the precise harmonic. However, these never fall on an integer value of x, except with specific z values (likely irrational) that lead to some specific integer x being in tune. Since we don't hit non-integer harmonics in music, if you played a random note, the probability of your fundamental note and any of its harmonics ever being perfectly in tune with any note is basically 0, even if you traverse multiple harmonics. What can you do with this information? You can say that mathematically, no one will ever be in tune. We're not quite done yet, as the title was "Mathematically Proving I'm Always In Tune." Remember, we said it depends on the definition of being in tune. But before that…

Pedantic Literature:

When I finished creating the first graph, I realized there could be more variables thrown into this. What if each octave is split into more parts? Let's set t as the number of equally spaced steps in an octave, also known as tone equal temperament (TET).

$t = \#TET$

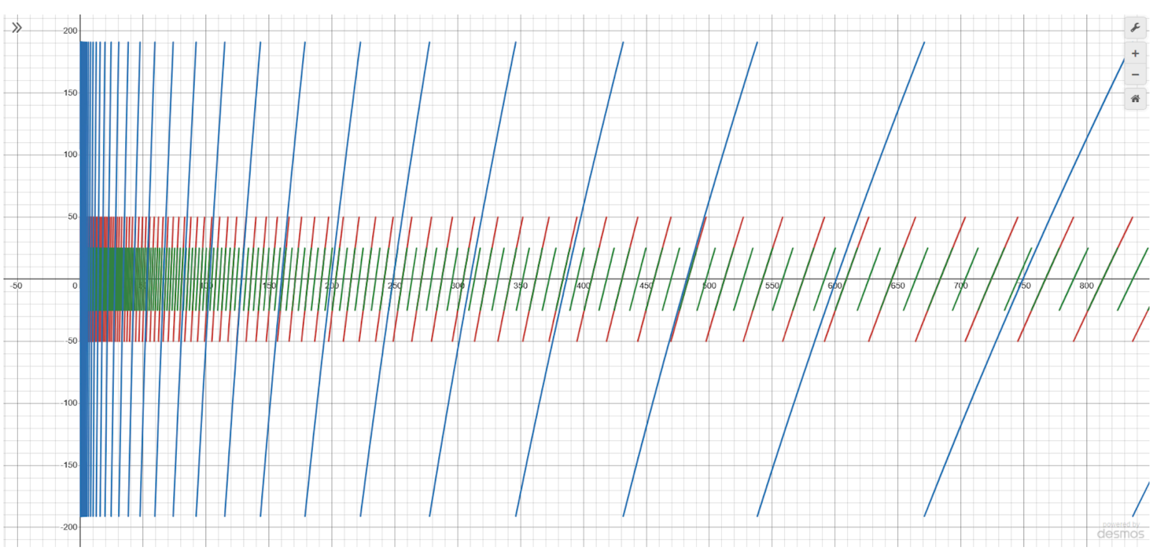
Replacing the 12's with t's from the previous equation gives us the new formula:

$$u = t\,log_2\left(\frac{zx}{r}\right)$$

$$f(x) = \frac{1200}{t}(u - round(u))$$

The only differences are that the 1200 cents were now divided into t-parts and that the semitones away part was altered into t-parts away. This can now create new divisions of an octave as shown below:

Red: t=12 Green: t=24 Blue: t=π



What can we do now? I was able to find a rare set of scales that considers an octave to have a frequency 3 times higher, rather than the conventional 2 times. To include these cases, let's set the variable o to represent the ratio of frequencies of an octave. We can now change the log_2's with log_o's, and throw in a $log_2(o)$ to account for the new number of cents in an octave.

$o = the\ ratio\ of\ frequencies\ of\ an\ octave$

$$f(x) = \frac{1200log_2(o)}{t}\left(t\ log_o\left(\frac{zx}{r}\right) - round\left(t\ log_o\left(\frac{zx}{r}\right)\right)\right)$$

This is a crazy formula with 5 independent variables. The harmonic series of music is much more complex and has sides that I have not touched on in this essay. But for now, let's complete the question we started with.

<u>The Final Stretch:</u>

What does it mean to be in tune? We figured out that to be exactly in tune with any note in any harmonic is impossible without superhuman precision that is more accurate than infinitesimal decimals. Simply, no one will ever be in tune with that definition.

However, there is a point where humans will not be able to tell if a note is in tune or out of tune. What if we created a graph that showed only the points where a note was within a certain cent difference? Well, here it is, our final formula:

$$u = t\ log_o\left(\frac{zx}{r}\right)$$

$$f(x) = \begin{cases} \frac{1200log_2(o)}{t}(u - round(u)) & \text{if, } \left|\frac{1200log_2(o)}{t}(u - round(u))\right| \leqslant c \\ undefined & \text{if, } \left|\frac{1200log_2(o)}{t}(u - round(u))\right| > c \end{cases}$$

$c = the\ allowed\ cent\ deviation\ from\ an\ "in\ tune"\ note$

If I considered just 5 cents off a note as still "in tune", this formula would show only the points where the note will be less than 5 cents off, sharp or flat.

For this to work in the real world, the harmonic number (x) must be an integer. Say I played a note at 450 Hz in the tuning of A=440 Hz, in a regular 12-TET with octave ratios being 2. For this one, we'll set the allowed "in tune" range to be 10 cents. With these values, I will be within the definitions of "in tune" on the 7th harmonic. However, there is a practical flaw in this. The 7th harmonic in this case corresponds to the note G. We probably want to find the note that is the one we originally wanted to play. In this example, the note we want would be A. With this new constraint that the note has to be the same as the starting pitch, we get… x=125. The actual harmonic we wanted was on the 125th harmonic, roughly 2.15 cents flat. Even though the note is at 56250 Hz, and well beyond human hearing, a household cat can hear a difference, going from 38.91 cents sharp to only 2.15 cents flat. What if we raise our standards? When c=1, we are "in tune" for the original note on the 1001st harmonic, being 0.42 cents flat. This also works with obscure variables. When: t=57, o=π, z=1234, r=e, and c=1, we would be "in tune" at the 968th harmonic. To find one more in tune, you can just tighten the constraints of c closer than before and get a better answer. This also means you will always be more in tune than anyone else just by restricting the value of c (when $c \in \Re$) as much as possible, and calculating the harmonic number. Ignore the fact that no one will be able to appreciate it since we can't hear these high frequencies.

Conclusion:

So what did we learn from this catastrophe of an equation? That you are in tune! In fact, more in tune than anyone else… ever! What can you do with that information? I suppose you can take a walk around the park knowing that no one else there is more accurate at singing one note, and that you are the definition of "in tune", for any practical definition of in tune… in the 927621st harmonic obviously.