

Mathematics Is Not a List of Rules

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April 2026

1 Introduction: The Art of the Ingredient

In the modern classroom, Mathematics has been transformed from a logical art-form into a largely procedural task. My journey through the A-Level curriculum has revealed a sobering reality: we are often taught to be calculators rather than thinkers. We follow "recipes" to solve quadratics or differentiate functions without ever questioning the ingredients. This creates a "procedural veil"—a barrier that allows us to find the right answer while remaining entirely blind to why that answer exists.

I want to lift this veil. Mathematics is not a list of separate rules; it is a unified landscape where even the most "abstract" complex numbers are rooted in physical geometry. By exploring four distinct proofs, I hope to show that when we stop treating math as a chore and start seeing it as a mystery, we find a world where logic and creativity are one and the same. Our journey begins at the very peak of mathematical beauty: a bridge between the real and the imaginary.

2 Euler's Equation: The Bridge of Taylor Series

Euler's Identity, $e^{i\pi} + 1 = 0$, is often called the "most beautiful theorem in mathematics." To understand it, we must first meet the catalyst of this equation: the imaginary unit i . In standard algebra, we are told that you cannot square a number to get a negative result. However, mathematicians defined i as the value such that $i^2 = -1$. Far from being "fake," i acts as a secondary axis that allows numbers to "turn" away from the standard number line and rotate through space.

The Components

Euler used **Taylor Series**—a method of turning complex, curving functions into simple, infinite sums of polynomials—to show how i links growth to rotation. These series represent how functions are made.

$$\begin{aligned}e^x &= 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots \\ \sin x &= \frac{x}{1!} - \frac{x^3}{3!} + \dots\end{aligned}$$

The Step-by-Step Derivation

We substitute $x = i\theta$ into the expansion for e^x .

$$e^{i\theta} = 1 + \frac{i\theta}{1!} + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \dots$$

Evaluating the powers of i ($i^2 = -1, i^3 = -i, i^4 = 1$), the real and imaginary parts naturally separate:

$$e^{i\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots\right) + i \left(\frac{\theta}{1!} - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots\right)$$

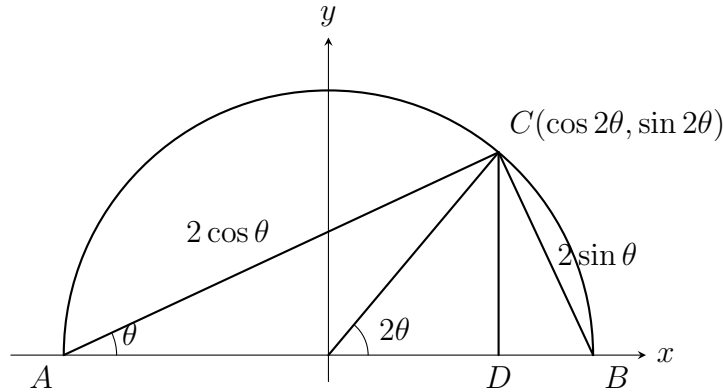
The two brackets are the Taylor series for $\cos \theta$ and $\sin \theta$. Thus, $e^{i\theta} = \cos \theta + i \sin \theta$. Substituting $\theta = \pi$ leads to $e^{i\pi} = -1 + 0$, or $e^{i\pi} + 1 = 0$.

Why does this matter?

This is surprising because it tells us that "imaginary" numbers are the secret language of rotation. Euler proved that exponential growth and circular rotation are two sides of the same coin. When we see $e^{i\pi}$, we are watching a point travel halfway around a circle in the complex plane to land exactly at -1.

3 The Geometry of Vision: Double Angles

If Euler's equation is a bridge of logic, then "Proofs Without Words" are the art of the landscape. This tradition, popularized by Roger Nelsen, uses visceral diagrams that allow our brains to interpret mathematics visually. We derive the Double Angle identities using a unit semicircle—a visual truth that requires no "recipes."



By using similar triangles (where shapes are scaled versions of each other) and the similarity $\triangle ACD \sim \triangle ABC$:

1. **Sine:** $\frac{CD}{AC} = \frac{BC}{AB} \implies \frac{\sin 2\theta}{2 \cos \theta} = \frac{2 \sin \theta}{2} \implies \sin 2\theta = 2 \sin \theta \cos \theta.$
2. **Cosine:** $\frac{AD}{AC} = \frac{AC}{AB} \implies \frac{1 + \cos 2\theta}{2 \cos \theta} = \frac{2 \cos \theta}{2} \implies \cos 2\theta = 2 \cos^2 \theta - 1.$

The Moment of Wonder

Why is this surprising? Because it reveals that these identities are not just strings of symbols. They are inevitable physical consequences of how lines intersect inside a circle. It transforms a math rule into a visual fact.

4 The Wave in the Algebra: Difference of Two Squares

The identity $x^2 - y^2 = (x + y)(x - y)$ is taught early on as an algebraic shortcut. To see it as a "winning" proof, we must link it to the trigonometry we just visualized. To do this, we use two tools:

- **Complementary Angle Identity:** $\cos B = \sin(\frac{\pi}{2} - B)$. This tells us that a cosine wave is just a sine wave shifted by 90 degrees.
- **Double Angle Identities:** The rules for $\sin 2A$ and $\cos 2A$ derived in the previous section.

Now you may be asking me : How on earth are these two concepts related? Well dear reader, I urge you to hold on, and wait for a moment.

The Standard vs. Trigonometric Proof

The standard proof is a statement: expand $(x + y)(x - y)$ to get $x^2 - y^2$. But if we replace x and y with coordinates $x = r \cos \alpha$ and $y = r \sin \alpha$:

$$\begin{aligned}x^2 - y^2 &= r^2(\cos^2 \alpha - \sin^2 \alpha) = r^2 \cos(2\alpha) \\&= r^2 \sin\left(\frac{\pi}{2} - 2\alpha\right) = r^2 \sin\left(2\left[\frac{\pi}{4} - \alpha\right]\right) \\&= 2r^2 \sin\left(\frac{\pi}{4} - \alpha\right) \cos\left(\frac{\pi}{4} - \alpha\right)\end{aligned}$$

Expanding via compound angle identities ($A = \pi/4, B = \alpha$):

$$\begin{aligned}&= 2r^2 \left(\sin \frac{\pi}{4} \cos \alpha - \cos \frac{\pi}{4} \sin \alpha\right) \left(\cos \frac{\pi}{4} \cos \alpha + \sin \frac{\pi}{4} \sin \alpha\right) \\&= 2r^2 \left(\frac{\sqrt{2}}{2}\right)^2 (\cos \alpha - \sin \alpha)(\cos \alpha + \sin \alpha) \\&= (r \cos \alpha - r \sin \alpha)(r \cos \alpha + r \sin \alpha) = (x - y)(x + y)\end{aligned}$$

Why this matters

This matters because it shows that the "Difference of Two Squares" is actually a specific snapshot of wave interference. It reminds us that behind every "flat" algebraic expression, there is a hidden, vibrating geometry.

5 Infinite Series: The Chocolate Bar Paradox

Our final stop takes us to the counter-intuitive world of the infinite. In A-Level, we learn $S_\infty = \frac{a}{1-r}$ via a subtractive trick: $S_n - rS_n = a - ar^n$. As $n \rightarrow \infty$, $r^n \rightarrow 0$, giving us the formula. But to find the wonder here, we must go to a chocolate shop.

The "Interesting" Way: Proof by Chocolate

Imagine a shop where 4 coupons = 1 free bar (which contains another coupon). If 1 coupon is worth V , then 1 coupon = $\frac{1}{4}(\text{bar} + \text{coupon})$. This creates an infinite chain: $\frac{1}{4} + \frac{1}{16} + \frac{1}{64} \dots$. To solve it logically: borrow a bar, take its coupon to make 4, and give them back. You have gained one bar and returned to zero coupons. Thus, 3 coupons equal 1 bar. Therefore, $1/4 + 1/16 + \dots = 1/3$.

The Story of Convergence

This is surprising because it provides a tangible intuition for infinity. It proves that an infinite sequence of actions can converge into a solid, finite result. It turns a formula into a story about fairness and debt.

6 Conclusion: Tearing the Veil

The derivations presented here demonstrate that mathematical identities are not merely rules, but intersections. When we view algebra through a trigonometric lens, or justify infinite series through chocolate, we gain a holistic understanding of the universe. We must stop viewing Math as a chore and start viewing it as a mystery. Only then can we see that the formulas are not the destination—they are merely the map to a much more beautiful reality.

If there's even one thing you takeaway from this essay it is this: Mathematics is not a list of rules.

References

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