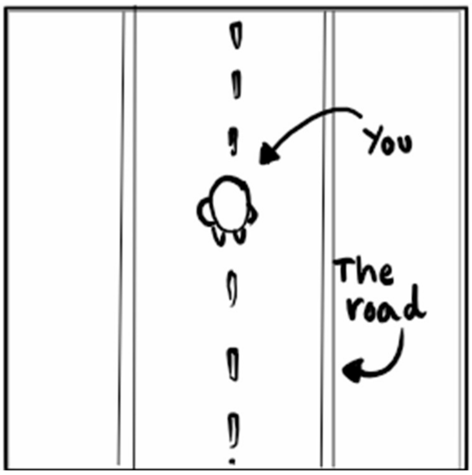
THE GEOMETRY OF SEEING

Eliana Gilbert-Heitler

Anyone who draws a lot, or who's ever tried to draw, will appreciate the skill required to convincingly portray space and form. You would think that those who excel in geometry - being masters of lengths, angles, areas and volumes - would find this task easier than those who are less mathematical. They may still struggle with tone and colour, but surely a mathematician would have acquired skills transferable into the world of art? In most instances, this is not the case. Why? Because the geometry we learn at school does not describe the same world we capture when we draw. The angles of a triangle or the length of a line are not concerned with where we are standing: they exist independent of us. When we draw, however, we portray what we *see*. So, I pose the question: Can we approach geometry in the same way? Is there a geometry that describes the world the way we see it?

To describe this geometry mathematically, we must first understand it, and the best way to understand anything is to experiment. So, imagine you are standing on a flat plane. You are in the centre of a straight road of constant width that goes on indefinitely.



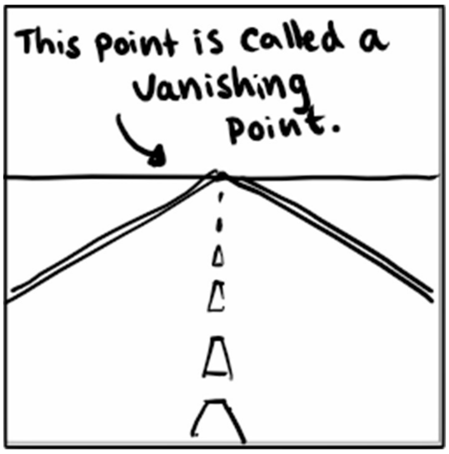
As children we are introduced to Euclidean geometry, so let us first consider the road from this familiar perspective. Viewed from above, the two sides of the road are clearly parallel since they are always the same distance apart. You will know from Euclid's parallel postulate that these two lines, being parallel, will never cross. However, standing on your imaginary plane, squinting into the distance, you see something that we have just defined to be impossible: the parallel lines *do* appear to meet. It seems like we will have to abandon Euclid, although we have found something that is worth investigating further.

The obvious thing to do now would be to walk down the road to the point where the lines intersect. So you walk, and walk, and after some finite amount of time you stop to check your progress. The elusive point is exactly where it was before. Being a mathematician you are unfazed, and consider the problem. There is some value x (the finite distance we have traveled towards the meeting point) and some quantity d (the total distance from the meeting point). We have found from our walk that no matter the value of x:

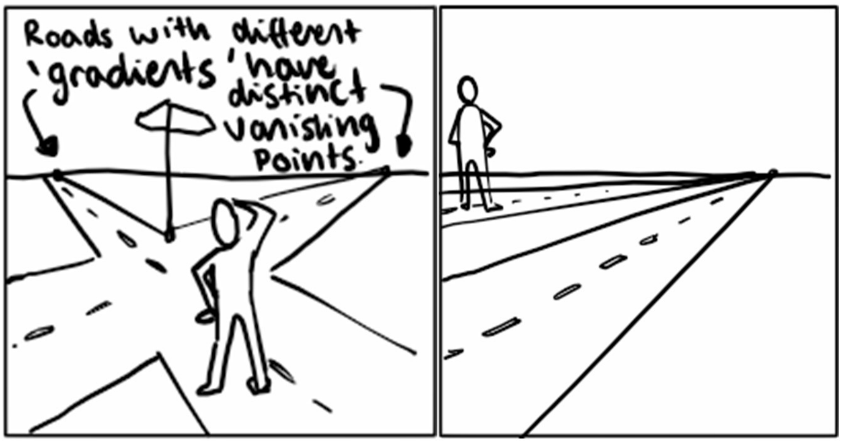
$$d - x = d$$

There is only one object that behaves this way: infinity. From this, we can gather that our road terminates at a point at infinity.

As a slight detour from our mission it would be helpful to compare our findings to ideas with which you might be more familiar. The point at infinity we have just discovered is conceptually no different from the infinity you encounter when summing a geometric sequence to infinity. In the latter case, infinity is used as a tool that allows us to reason about processes that never terminate. We cannot sum to infinity – that would take infinite time – but if we use our abstract concept of infinity as a framework, we can reach a finite answer. Similarly, we can't walk to the point where an infinite road ends. However, we can find the precise point in our field of vision where it appears to end by stitching an infinitely far away point into the very fabric of our world.



Back to our imagination. Is our road special? Have we just stumbled upon the single magical road that leads to infinity? The advantage of being in an imaginary scenario is that we can simply build more roads to investigate further. Let's say our initial road is straight ahead. If we build another road parallel to ours, we will see that it appears to terminate at the very same point. Now let us build another road heading off at a diagonal. This road too has a vanishing point, but at a different point on the horizon. In fact, every road heading off in a unique direction will have a unique vanishing point. The locations of these points are not random, of course. They are entirely determined by the road's direction.



Having gathered this evidence, let's think through what we know. We will call our geometry the real projective plane, denoted as $P^2(R)$. As we have just discovered, we can break this space down into the standard plane with which we are familiar from Euclidean space (R^2) with a line at infinity ($P^1(R)$) stitched onto the horizon:

$$P^2(R) \ = R^2 \cup P^1(R)$$

The 'line at infinity' is clearly made up of all the vanishing points at the end of every possible road in the plane. Conceptually, it may be helpful to think about this line at infinity as a circle of radius infinity that contains the plane we are standing on.

You may have noticed that until this point I have intentionally restricted us to only consider things within our field of view. So let us consider what lies out of our view. Stand up in your imaginary plane. Look left, look right, and over your shoulder. Is that another vanishing point behind you? Well, not exactly. In fact, this vanishing point and the one in front of you are one and the same. Let me explain.

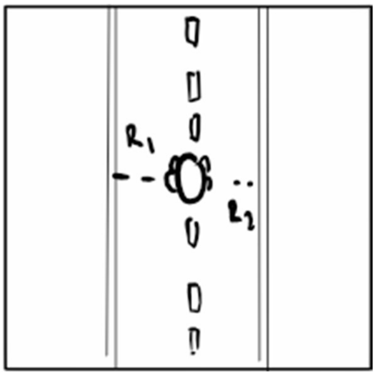
Stand in the centre of the road and look straight down its length in one direction. It just so happens that you are standing in a glass cylinder and you have two markers in your pocket, green and blue. We are going to use the walls of this cylinder and our markers to map the entire 360° horizon onto points in our field of vision. On the glass wall directly in front of you, draw a green dot where the vanishing point appears to be in your field of vision (G_1). Do the same behind you, this time drawing a blue dot (B_1). Now turn around 180° and do exactly the same thing as before. Label the vanishing point in front of you green (G_2) and the one behind blue (B_2).

Before we go any further, I'd like to remind you that we are building a geometry that describes the way we *see*. Everything is relative to us. This means that when we turn ourselves 180°, the whole coordinate system will flip as well, so our green dot is always marking 'positive' infinity and blue 'negative' infinity.

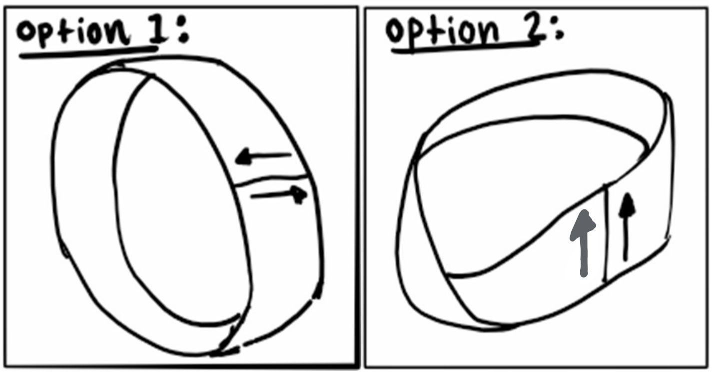
Back to our dots. You will find that B_2 is in exactly the same place on the glass as G_1. Similarly, G_2 is in the same place as B_1. From our vantage point, therefore, the two vanishing points are entirely interchangeable. We could easily consider the infinity in front of us to be 'negative' and the infinity behind us 'positive'; we could also consider them both to be 'positive' or both 'negative'. In other words, I could blindfold you and spin you around a hundred times, and when you opened your eyes you wouldn't be able to tell which way you were facing, nor would it matter. In fact, defining both vanishing points as the same point describes better how we experience these points. There is no invariant geometric property that can distinguish them, so why distinguish them at all? I hope you can see the significance of this. We are letting go of yet another Euclidean inclination: the idea that orientation is absolute.

I previously described the plane as being enclosed by a circle of radius infinity. But if antipodal points on this circle are actually the same point, as we have discovered, this geometric description becomes more complicated. Clearly, we need to ‘glue’ all these opposite points together.

Before we do that, it would be helpful to label some things. Grab some chalk and draw a line across the road where you are standing. This splits our road into two parts: R_1 (ahead of you) and R_2 (behind you).



Now jog an almost infinite distance down R_1 and draw an arrow pointing right. Do the same for R_2. Then jog back to the dividing line. From where you stand, the two arrows you have just drawn are facing in opposite directions. Now let us imagine pulling together these two opposite points. For the sake of simplicity, we can ignore the rest of the plane for now, and just imagine the road being bent over itself to join the two sides. We want to attach the two points such that the road is a seamless loop - a loop that just happens to run through a point at infinity. Clearly, we have two options:



The key here is to consider the more general result of our ‘dots on glass’ experiment. Instead of drawing dots over the vanishing points, we could have drawn dots over any two points down the road from us in either direction. Unsurprisingly, we would have come to the same conclusion as before: the two points are exactly the same. Clearly, this implies that walking forwards down R_1

and walking forwards down R_2 are equivalent. All we have to do now is decide which of our two options pictured above achieves this equivalence.

For this final thought experiment it will be helpful to have two people, so I will join you in our imagined plane. You walk down R_1 while I walk in the opposite direction, down R_2. If we both walked an infinite distance at the same speed, we would arrive at the vanishing point at exactly the same time. At this point, you - walking forwards - must see exactly what I see, since our walks are interchangeable. That means that my right is your right, rather than your left. This might go against your intuition, so just remember that we have not simply walked in opposite directions around some loop and met on the other side. We have walked in two versions of the same direction and met at the same point. This isn't a space we can ever really physically encounter, so it would be understandable if your instincts were misleading. Returning to our two options, the second is clearly the correct version of the road since my right side and your right side (represented by the arrows) are the same direction.

You may recognize option 2 as closely resembling a mobius strip. This is a one-sided and non-orientable surface constructed using a half twist. Importantly, while the road itself is non-orientable, if we imagine walking around this road our local orientation is consistent throughout the journey. Our understanding of left and right does not just switch at a magical point at infinity.

A final challenge: try to visualize the structure of the entire plane for yourself. All you have to do is construct a mobius strip with every single road in the plane simultaneously. Easy, right? Actually, this surface is actually impossible to embed into 3D space without self-intersection (essentially tearing the mesh of space). This is not to say that the projective plane is not mathematically consistent. It is. The plane is just not something that we have the ability to ever fully visualise, just like four-dimensional space.

We have reached the end of our exploration. I started by posing the question: Is there a geometry that describes the way we see? Following logic, we discovered an unintuitive landscape where opposite directions are the same direction and every road returns to itself with a half twist. In trying to understand mathematically how we see the world, in other words, we were forced to imagine a space that we can never truly see at all. If artists have for centuries been navigating this paradox with ease, then perhaps to be an artist is simply to be a mathematician who prefers a brush to a compass.