Why Do Spheres Occur So Frequently in Nature?

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1. The Thought Experiment

So, one day while having a conversation about physics with my friend, he asked a really deep question: *“Why do spheres occur so frequently in nature?”* If we think about it, we start noticing how often spherical or sphere-like objects appear—planets, soap bubbles, ground state orbitals of hydrogen atoms, and so on. This question quickly leads into deeper mathematical territory, because at first it is not even clear where to begin or what kind of approach one should take.

Let us start by simplifying things with a thought experiment. Imagine a universe in which space has no preferred direction, so that the laws governing physical processes remain unchanged under rotations. Now fix a point in such a space and consider another point at some fixed distance away, where an effect is observed to propagate from the first point.

If we now change the orientation of this second point while keeping the distance fixed, rotational invariance implies that nothing physically observable should change. The universe does not distinguish between different orientations, so all such rotated configurations are equivalent. Thus, the effect of the cause cannot depend on direction, but only on distance. In this setting, the natural geometric structure that appears is a sphere: the set of all points at a fixed distance from a given point. This suggests that whenever a propagation law is invariant under rotations, spherical symmetry naturally emerges in its behavior. More precisely, spherical symmetry means that the system depends only on distance from a point, not on direction.

From this thought experiment, we can make two key observations. First, the appearance of spheres in nature seems to require that the underlying laws of propagation exhibit spherical symmetry. Second, rotational invariance appears to be the deeper assumption behind this symmetry. However, the precise logical connection between the two is still not fully clear.

This brings us to a reformulation of the problem: if a law is rotationally invariant, can we show that it must necessarily be spherically symmetric, in the sense that its behavior depends only on distance from a point and not on direction?

2. Expressing The Idea Mathematically

So far, our discussion has been largely intuitive. To proceed further, we now express these ideas mathematically.

A natural example of a spherically symmetric law is Newton's law of gravitation:

$F(r) = \frac{GMm}{r^2}$ Here, the force depends only on the distance between two bodies and not on the direction, reflecting spherical symmetry. This already hints at a connection between physical laws and symmetry. To generalize this idea, let us describe the propagating quantity by a function u(x), where $u: \boldsymbol{R}^3 \to \boldsymbol{R}$ and $x \in \boldsymbol{R}^3$ and $|x| = r$. We restrict ourselves to scalar fields for simplicity. In physics, such quantities are typically governed by differential equations, like motion, waves and heat, we therefore assume that u(x) satisfies a differential equation of the form:

$$L[u] = 0$$

where L is a differential operator. In many physical situations, L is linear, a property that ensures solutions behave in a structured and predictable way.

Now we have made a fair bit of progress towards expressing our idea mathematically, but in order to describe rotational invariance we need a systematic way to describe rotations mathematically. So how do we do that?

3. Prelude to The SO(3) Group

There are several ways to describe rotations in three dimensions; however, for our purposes, we adopt the matrix representation. In particular, we consider rotation matrices acting on $\boldsymbol{R}^3$

$$R_x(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$R_y(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$

$$R_z(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We begin by defining the three principle rotations about the coordinate axes, denoted by

$R_x(\theta), R_y(\theta), R_z(\theta)$ corresponding to rotations about the x-, y-, and z-axes respectively. Each of these matrices generates rotations about mutually perpendicular axes. By composing these rotations, we can generate more general rotations. For a vector v, consider:

$$v' = R_z(\gamma)R_y(\beta)R_x(\alpha)v$$

.

where the order matters, since rotations do not commute. Now this sequence can be used to describe any general rotation about a particular axis. Thus:

$$R = R_z(\gamma)R_y(\beta)R_x(\alpha)$$

Such transformations preserve lengths and angles, and are characterized by matrices R satisfying:

$$R^T R = I, \det(R) = 1$$

4. The SO(3) Group

We define the set of all such rotation matrices by

$$SO(3) = \{R \in \boldsymbol{R}^{3x3}\colon\ R^T R = I, \det(R) = 1\}$$

Under matrix multiplication, these transformations combine to form a group, known as the special orthogonal group in three dimensions. This structure provides the natural mathematical language for describing rotational invariance.

5. Setting Up the Goal

Now that we have a mathematical language for rotations, we can restate our main question more precisely: does rotational invariance of a law imply spherical symmetry of the resulting behavior?

We assume the governing law is given by a differential equation:

$$L[u(x)] = 0$$

Rotational invariance means that if u(x) is a solution, then any rotated version of it is also a solution. That is,

$$if\ L[u(x)] = 0\ then\ L[u(Rx)] = 0, \forall R \in SO(3)$$

Next, we express spherical symmetry. A function is spherically symmetric if its value depends only on distance from the origin, so that orientation is irrelevant:

$$u(x) = u(Rx), \forall R \in SO(3)$$

This further implies $u(x) = f(r)$ where $|x| = r$

So now our goal mathematically stated becomes precisely:

$$If\ L[u(x)] = 0\ and\ L[u(Rx)] = 0,$$

$$\forall R \in SO(3)\ for\ a\ solution\ u(x)\ then\ is\ it\ necessary\ that\ u(x) = f(r), |x| = r\ ?$$

And if not, what additional conditions are required for this to hold, and do such conditions naturally arise in physical systems? In the next section we prove that under two key mechanisms occurring in nature, rotational invariance of the equation forces the existence of a spherically symmetric solution, since rotational invariance alone isn't strong enough to imply that solutions must be spherically symmetric.

6. The Proofs Under Suitable Mechanisms

If u(x) is a solution to $L[u(x)] = 0$, then under the action of SO(3) it generates an orbit, namely the collection of all rotated versions of u(x):

$$O_{SO(3)}\big(u(x)\big) = \{u(Rx) \colon \forall R \in SO(3)\}$$

By rotational invariance, each of these is also a solution. Thus, instead of a single solution, we obtain an entire family of solutions related by rotation, given by the orbit.

Mechanism I:

Many differential equations in nature have a useful property: under suitable boundary conditions, they admit a unique solution. A simple example is Newtonian mechanics, where specifying initial conditions determines a unique trajectory for the differential equation of motion. So for this mechanism, we assume that $L[u(x)] = 0$ has a unique solution for rotationally invariant boundary conditions, meaning that if u(x) and v(x) are solutions, then u(x) = v(x). Since each element of the orbit is also a solution satisfying the same conditions, uniqueness leaves no alternative but for all of them to coincide. Thus, the entire orbit $O_{SO(3)}(u(x))$ must collapse to u(x).

This implies:

$$u(Rx) = u(x), \forall R \in SO(3)$$

and hence the solution is invariant under rotations. Therefore, u(x) depends only on the distance from the origin, so u(x) = f(r), establishing spherical symmetry. This shows that a spherically symmetric solution always exists if we have uniqueness of solutions.

Mechanism II:

In many physical systems, the observed configuration is not just any solution of the governing equations, but one that minimizes a certain quantity, often interpreted as energy. For instance, a stretched string settles into a shape that minimizes its elastic energy, and a soap film arranges itself to minimize surface area.

This idea is formalized through variational principles: instead of solving a differential equation directly, one considers a functional, $E[u(x)]$ which assigns a number (energy) to each configuration u(x), and seeks the function that minimizes it. The governing differential equation then arises as a condition for this minimization, where the minimizer is a solution to the differential equation.

Formally the differential equation arises from:

$$\frac{\delta E}{\delta u} = 0 \; if \; u \; is \; a \; minimizer \; of \; E[u]$$

This condition works almost synonymously with finding maxima and minima of functions.

Now suppose the energy functional is rotationally invariant, so that

$$E[u(Rx)] = E[u(x)], \forall R \in SO(3)$$

Then all rotated versions of a minimizer have the same energy.

To understand how symmetry can still emerge even without uniqueness, we turn to another idea that appears naturally in many physical systems: convexity.

A function is called convex if the value at an average is no greater than the average of the values. The same idea extends to functionals. A functional $E[u]$ is convex if, for any two configurations u and v,

$$E\left[\frac{u+v}{2}\right] \leq \frac{1}{2}E[u] + \frac{1}{2}E[v]$$

Or more generally

$$E\left[\frac{u_1 + u_2 + \cdots + u_n}{n}\right] \leq \frac{1}{n}(E[u_1] + E[u_2] + \cdots + E[u_n])$$

The basic idea is

$$E[average \; of \; configurations] \leq average \; of \; E[u] \; for \; configurations$$

Convex functionals are not artificial constructions; they appear naturally in many physical systems. In problems appearing in electrostatics, steady-state heat distributions and elasticity, the energy functionals appearing are convex in nature.

Now in this mechanism we choose that our governing differential equation comes from a variational principle and the underlying functional is rotationally invariant and convex in nature.

For a minimizer u(x) an orbit is generated again:

$$O_{SO(3)}\big(u(x)\big) = \{u(Rx) : \forall R \in SO(3)\}$$

Right now all the orientations in the orbit are scattered, and there is not much connection between them other than, they are all minimizers. Thinking a bit, to make further progress we need something which will also be a solution but somehow bring all the information of the different configurations in the orbit together, in the same spirit as Mechanism I.

So define

$$u_{avg}(x) = \frac{\int_{SO(3)} u(Rx)d\mu(R)}{\int_{SO(3)} d\mu(R)}$$

Where we average over all possible configurations in the orbit. Intuitively, averaging smoothens out directional irregularities by distributing them evenly across all orientations, combining the information from the entire orbit into a single function u_{avg}

Now we need some weightage given to every configuration in order to average, which would ultimately serve as a differential to integrate over. Also this assigning of weights should not bias between two different orientations, to preserve the notion of rotational invariance.

Here $d\mu(R)$ is the differential which serves this purpose. It assigns a notion of equal weightage to all the rotations in SO(3) which allows us to integrate over the configurations of u. it has two key properties which is very useful here:

$$for\ Q \in SO(3), d\mu(QR) = d\mu(R)$$

QR refers to some other rotation in SO(3) but since it gives equal weight to every orientation the weight doesn't change.

Secondly,

$$\int_{SO(3)} d\mu(R) = 1$$

which means to say that sum of every possible weights of rotation in SO(3) is 1.

Which gives $u_{avg} = \int_{SO(3)} u(Rx)d\mu(R)$

Now combining convexity and rotational invariance, we get

$$\begin{aligned} &if\ \ u(x)\ is\ a\ minimizer\ of\ E[u(x)]\ then \\ &E\big[u_{avg}\big] \le \frac{\int_{SO(3)} E[u(Rx)]d\mu(R)}{\int_{SO(3)} d\mu(R)} = \int_{SO(3)} E[u(Rx)]d\mu(R) \end{aligned}$$

But since $E[u(x)] = E[u(Rx)], \forall R \in SO(3)$

$$\int_{SO(3)} E[u(Rx)]d\mu(R) = E[u(x)] \int_{SO(3)} d\mu(R) = E[u(x)]$$

Thus

$$E[u_{avg}] \leq E[u]$$

Which proves that if u is a minimizer then u_{avg} will also be a minimizer and thus a solution to the differential equation.

Now we will prove that $u_{avg}$is spherically symmetric

Consider

$$u_{avg}(Qx) = \int_{SO(3)} u(QRx)d\mu(QR)\,, for\ any\ Q \in SO(3)$$

Now applying Q on R just permutes SO(3), that is, it changes the order of rotations, but that doesn't add or subtract anything, thus the group as a whole remains unchanged, therefore there is no change in domain of integration.

Let S = QR then $d\mu(S) = d\mu(R)$

Thus

$$u_{avg}(Qx) = \int_{SO(3)} u(Sx)d\mu(S) = \int_{SO(3)} u(Rx)d\mu(R) = u_{avg}(x), \forall Q \in SO(3)$$

Implying $u_{avg}(Rx) = u_{avg}(x), \forall R \in SO(3)$ so u_{avg} depends only on the distance from the origin, so $u_{avg}(x) = f(r)$, establishing spherical symmetry. This essentially shows that a spherically symmetric solution to the differential equation exists, if we choose to average over the orbit.

7. Conclusion and Reflections

So this is the section where I talk about what we have achieved, and what we haven't, and reflect on with a bit of honesty. We started with this simple question about spheres occurring so frequently in nature, followed by a thought experiment, and then just by going with its spirit, and trying to express things mathematically lead us to such elegant mathematics. At each mechanism discussed that every assumptions wasn't just a mere step, it was motivated by actual phenomena appearing in nature. Showing that spherical symmetry of solutions arising from these mechanisms under rotational invariance was a very elegant mathematical exposition that we undertook, but by doing this we definitely didn't answer spherical symmetry as a whole universal occurrence. What we did show, was that if certain conditions are met spherical symmetry will always be one of nature's option, which solves a certain subset of our question. Now why does nature specifically obsess over the spherically symmetric solutions? Well that is a question about dynamical systems and stability, which was honestly out of scope for this paper.

But nonetheless, this whole journey was one of the examples why I personally like mathematics so much, because how seemingly out of nowhere a philosophical question about spheres links with symmetry group theory and differential equations when we look with a mathematical lens. I would like to acknowledge insightful discussions with my teachers, as well as the friend whose question inspired this investigation. Down below I will be sharing a few references for curious people who want to dig more into this.

References:

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