Maths essay competition entry

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1’s and oh’s!

The subject of this essay is a puzzle that I think has a wonderful solution, and tells an almost anecdotal tale in problem solving. I stumbled upon this puzzle when I was bored in maths class once, and I started writing random things in my exercise book. One of those things was a really long string of 0’s that took up the length of the page, with one of the zero’s replaced with a 1. My idea was that I would create a new list below it, by putting a number underneath and in between the two numbers above it, defined by the difference of the 2 numbers. For example, if the first 2 numbers in my sequence were a 0 and a 1, then I would put a 1 (1-0) below the 0 and 1. I would do that for every pair of adjacent numbers, to create a new sequence that is 1 digit shorter. I would then do this process again with the new sequence, and again and again and again until I am left with just one digit at the bottom of an upside down triangle.

My question was this: by looking at a long string of 1’s and 0’s, can you determine what the final digit will be (a 1 or 0) without computing the whole triangle? Its complicated to explain, but fairly simple when you think about it.

Often in problem solving, you are told to try a simpler case than your original problem, and so, quite naturally, I did. One digit. Its just… what the digit is. Too simple. What about… 3 digits? Let’s see, remember there is only one 1, and the rest are zeroes. If the 1 is on the end the final digit will be a 1, and if its in the middle, it’s a 0. (Keep that in mind, if it’s on the end it’s a 1) Um… maybe there’s a pattern? Here, what about 4 digits. Okay, um… oh, wherever you start with the 1, the result will always be a 1.

Just by doing this over and over again, it can be hard to spot a pattern or anything that links the sequence and the final digit. Here is where we come to our first lesson in problem solving: simpler doesn’t always mean small numbers. Don’t get me wrong, most of the time, it does, but sometimes, inputting cases that aren’t necessarily small, but simple or could lead to an insight are much more helpful, than looking at what happens when n is equal to 1 or something.

In our case, the option we're looking for… is infinity. What this means will be explained later, but intuitively it makes sense, if every example is contained within the example in which we had an infinite triangle, then why not just see if there's a pattern in said triangle? And if you then say: "oh, but that violates the rule of it being simple! Isn't that literally the most complex it could get?". To which I say… shush. We're getting to the good bit, so just, trust me okay?

The way we are going to map out an infinite triangle is imagine an infinite string of 0's, a 1, and then another infinite string of 0's. Now follow the same rules as before. Take the difference of two numbers above to get the number below. So on the second line there will be an infinite string of 0's, then two 1's, then an infinite string of 0's. Then on the third line it would be an infinite string of 0's, then 1 0 1, then an infinite string of 0's. If you carry on this pattern you get an infinite number of lines which will contain every possible finite triangle which starts with a top line with only one 1 in it. Therefore every triangle I imagined above is contained within this infinite pattern.

With this the "right way up" triangle (the digits that the top 1 effects – that is our finite triangle) can get as large as it like. Try this yourself, it's a good time consumer if you're bored (you don't need infinite paper, you can just imagine the 0's are there). I actually tried this, very early on in the process of me trying to solve this problem. But I couldn't see a pattern, at all. After many days, and countless minutes of frustration (not hours, I'm not *that* dedicated), I finally found something. I was trying to see if this triangle (the right way up one), shows up in any other places.

My train of thought was "*what's a very famous triangle*? *Um, Pascal's triangle?"*, and so I looked it up. I copied it down and started circling the places where 1's are in my triangle, and tried to spot a pattern in Pascal's triangle. This is when I had that key insight, that "oh!" moment you need to solve any puzzle. "*It's the odd numbers!"* I exclaimed excitedly. But… why? I started to form an intuition in my head, but before that, I needed to solidify my belief, so I kid you not, I checked up to the 50th row in Pascal's triangle before I allowed myself to believe what I had just discovered.

The way to think about why this is true, is to reimagine Pascal's triangle to not start with a 1, but with an infinite string of 0's, a 1, and then an infinite string of 0's again. Then, you just use the same rules as normal, the 2 numbers above add together to make the number below. What you get is just normal Pascal's triangle, but with a bunch of 0's on either side.

Now imagine replacing every even number with an E, and every odd number with an O. What we get is basically identical to our original infinite triangle with just 0's and 1's. This is because the rules for making the triangles are exactly the same. (E+E=E, 0-0=0… etc.). For example, adding together 2 odd numbers on Pascal's triangle is the same as subtracting a 1 from a 1 on our triangle.

What we have just proved can be difficult to relate back to our original problem, but it basically means: imagine the infinite sequence of 0's with a 1 in it, and that that 1 is the tip of Pascal's triangle and then complete Pascal's triangle. Then draw a box around your string of numbers with the 1 in the correct position (for example 0 1 0 or 1 0 0) and complete your triangle by using that box as the top of a new triangle. You can determine what the last digit will be of your original sequence by looking at the number in the bottom tip of the triangle we just created and seeing if it is odd or even in the Pascal's triangle (with odd=1, even=0).

Just as a quick sanity check, lets see if our observation that if the 1 is on the end, then our final digit is always a 1, makes sense with our new strategy. And it does! This is because if you take that 1 to be the tip, then as you follow the edge down to get to the final digit, it will always be a 1. Now you may be wondering: *"but that doesn't solve the puzzle at all! We've just found an arguably less efficient way to calculate that ending digit"*, to which I cleverly respond… shut up! You always ruin everything. Ugh.

By the way, if my explanations aren't clear, or I was genuinely wrong, PLEASE tell me. Also, do I win the "most mentions of the word *triangle* award?"