How powerful is a compass, really?

When I was younger, drawing circles felt deceptively easy. A quick flick of the wrist, and a masterpiece was formed. In reality, it was far from perfect: it bulged on one side, flattened on the other, and resembled more a lopsided potato than anything geometrical.

Then, one day, we were introduced to the miraculous solution to all our problems: the compass, consisting of two rigid arms and an intimidatingly sharp tip (somehow managing to prick me a thousand times) it formed the perfect circle, again, and again, and again.

And at that moment, it felt like the key to constructing any shape perfectly. Little did I know, this wasn't always the case.

1) What can a compass do in mathematics?

To understand how a compass works in geometry, we must travel back to ancient Greece, around 300BC, when Euclid compiled one of the most influential texts in mathematical history: Elements. In this text, Euclid established the fundamental principles for constructing shapes, with just two tools: the compass and straightedge.

With these surprisingly simple instruments, mathematicians constructed circles, bisectors and regular polygons, all with impressive precision. These constructions were guided by a set of rules outlined in Euclid in Elements, some of which are summarised below.

1) The straightedge allows a straight line to be drawn between two points, and can be extended indefinitely in a straight line
2) A circle can be drawn with any centre or radius

2) Trisecting the Angle

Ultimately, mathematicians discovered that not all constructions are possible using a compass and straight edge, one being French mathematician Pierre Wantzel, who proved in 1837 that trisecting an arbitrary angle using these classical tools is impossible.

To understand why, consider trisecting a 60° angle. If the trisection was possible, we would be able to construct an angle of 20°.

Using the triple angle identity, let $x=\cos(\theta)$

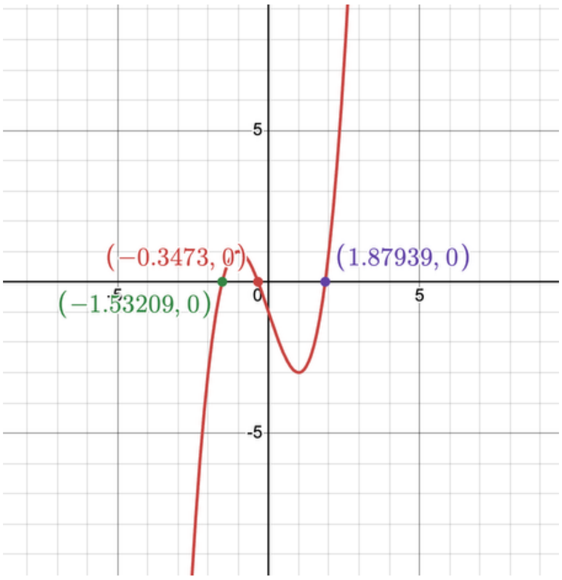
$$\mathbf{Cos\ (3\theta) = 4cos^3\ - 3cos\ \theta}$$

And setting $\theta=20°$, we can expand and obtain the cubic

$$\mathbf{4x^3 - 3x - \tfrac{1}{2} = 0}$$

Now, to simplify the equation further, we can let $2x=y$, giving us

$$\mathbf{y^3 - 3y - 1 = 0}$$



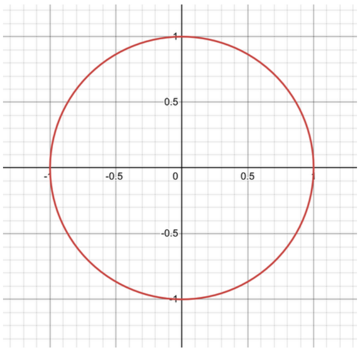
If 20 ° were constructible, then the value of y would also have to be constructible. Now, using the rational root theorem we know that any rational solution must be a factor of the constant term divided by a factor of the leading coefficient. The only possibilities are 1 and –1, and neither satisfy the equation. Therefore, the cubic has no rational roots and cannot be factored using rational numbers.

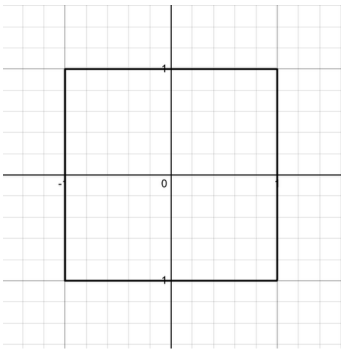
Hence, since there is no rational root to this equation, 20° is not constructible so a 60 ° angle cannot be trisected using these tools.

3) Squaring the Cube

Now let's move onto squaring the circle. If it were possible, we would be able to find the area of a circle equal to the area of a square.

First let's consider the general area of a circle, radius r, as well as the general area of a square of side length x.





$A=\pi r^2$
$A= x^2$

Now, we have formed 2 simultaneous equations which we can solve. When we equate the two equations and solve for **X** we obtain:

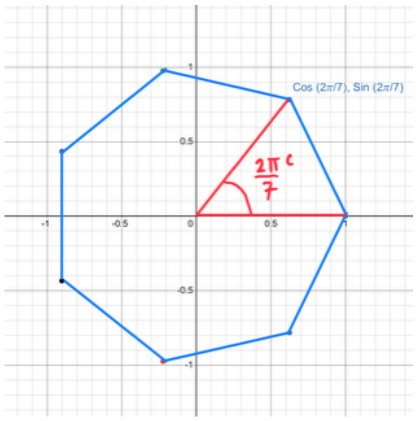
X=r $\sqrt{\pi}$

Finally, we can conclude that since π is a transcendental number, it can't be the root of a non-zero polynomial with rational coefficients, therefore it is not algebraic. Consequently, it is impossible to construct $\sqrt{\pi}$ using a straightedge and compass, because it too is not algebraic.

4) Constructing a Regular Heptagon

To prove that you cannot construct a regular heptagon using a compass and straightedge, we can assume without loss of generality, that the heptagon is centred at the origin.

To move from one vertex to another, we must be able to rotate an angle of $2\pi/7$ radians.



As aforementioned, a straightedge allows a straight line to be drawn between 2 points, and a compass allows a circle of any radius to be drawn. Therefore, since we can only draw straight lines and circles using these mathematical instruments, any point on a constructed shape must be a solution to a straight line or a quadratic.

This means we can only have solutions which are square roots or repeated square roots, which are only solutions to polynomials whose degree is power of 2.

Now, we must prove that cos $2\pi/7$, cannot be the solution to a polynomial of degree of power of 2. To do this, we can construct the seven roots of unity. Since by constructing a regular polygon, we are effectively equally spacing points on a circle, we can let those points be

z= e^$2\pi i/7$

Since z satisfies the equation

z^7 =1

We can then rearrange to get

z^7-1= 0

Now, using the standard entity for n=7

$z^7-1=(z-1)((z^6+z^5+z^4+z^3+z^2+z+1)$

We can expand this to get

$z^7+z^6+z^5+z^4+z^3+z^2+z+(-z^6-z^5-z^4-z^3-z^2-z-1)$

Since most of the terms cancel out, we are left with

z^7-1

Now we know that $(z^6+z^5+z^4+z^3+z^2+z+1)=0$

We can divide through by z^3 to get

$z^3+z^2+z+1+z^{-1}+z^{-2}+z^{-3}$

And group the terms into pairs

$(z^3+z^{-3})+(z^{-2}+z^{-2})+(z+z^{-1})+1=0$

Now we can let $t=z+z^{-1}$

Using our definition for t, we can derive the 3 identities needed to prove that cos $2\pi/7$ is a solution to a cubic equation.

Since $t=z+z^{-1}$, we can square both sides leaving us with

$t^2=z^2+2+z^{-2}$

Then we can rearrange to get

$t^2-2=z^2+z^{-2}$

Through a similar process, we can get the other identity of $t^3-3t=z^3+z^{-3}$

$t^3=z^3+3z+3z^{-1}+z^{-1}$

Then, we can group the terms and replace $z+z^{-1}$ with **t** to get

$t^3=(z^3+z^{-3})+3t$

So

$t^3-3t=(z^3+z^{-3})$

Then, we can substitute the identities for **t, t^2-2 and t^3-3t** so that out equation becomes

$(t^3-3t)+(t^2-2t)+t+1=0$

So further simplification gives us the cubic equation

$$t^3 + t^2 - 2t + 1 = 0$$

Finally, we can compute **t**, using our defined **z**

$$t = z + z^{-1}$$
$$t = \cos 2\pi/7 + i\sin 2\pi/7 + \cos 2\pi/7 - i\sin 2\pi/7$$
$$t = 2\cos 2\pi$$

We can conclude that 2 cos $2\pi/7$, satisfies a cubic equation, and hence cos $2\pi/7$ must also satisfy a cubic equation. Since constructible numbers must have a degree of power of 2, and 3 is not a power of 2, a regular heptagon cannot be constructed using a compass and a straightedge.

Conclusion

So yes, the compass and straightedge, have forever been portrayed as geometry's best friends, but even they can't convince every shape to appear on the page. Some stubborn shapes just refuse to be drawn (yes, I'm talking about you heptagon...).

But, before you abandon that compass and straightedge completely, remember geometry isn't all doom and gloom! In fact, the proofs we've explored today are the perfect conversation starters.

Next time you're at a party, challenge a friend to cut the cake into 3 perfect slices; you're bound to spark a heated debate!

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