The Frobenius Coin Problem through McNuggets

In this essay, we will explore the Frobenius coin problem with a linear Diophantine equation through the lens of McNuggets and some other fundamental aspects of Number Theory such as arithmetic operators to better understand the applications of this area of mathematics.

Arithmetic Operators

Arithmetic operations are the fundamental, functional methods of combining, manipulating, or transforming numbers in mathematics. Examples of these include addition and division. This section will develop our understanding of different arithmetic operators allowing us to solve more problems inside of number theory.

One common arithmetic operator is greatest common divisor (GCD). The GCD of two numbers is the largest number that can divide the two original numbers.

What is the largest container that could measure exactly both 403 litres and 744 litres?

We first find the prime factors of 403 and 744.

$$403 = 13 \times 31$$

$$744 = 2 \times 3 \times 31$$

What is the largest number that can divide both 403 and 744? We could fill the 403-litre tank using a 31-litre bucket by pouring it 13 times. You could fill the 744-litre tank using that same 31-litre bucket by pouring it 24 times. Since 31 is the largest factor they both share, a 31-litre bucket is the biggest one that measures both perfectly.

Another commonly used arithmetic operator is the lowest common multiple (LCM) is the smallest possible integer that is divisible by two or more given numbers.

Ronald McDonald has two chicken nugget guns; he fires one every 6 minutes and the other every 8. They both fire at 9am, so when do they next fire together?

We first find the prime factorisation of 6 and 8.

$$6 = 2 \times 3$$
$$8 = 2^3$$
$$LCM = 2^4 \times 3 = 24$$

So, they next fire together at 9:24am

Using these two operators unlocks a larger mathematical toolbox to solve increasingly more complex problems in number theory and allows us to understand the solution to the Frobenius Coin Problem.

What is a linear Diophantine equation?

A linear Diophantine equation is a polynomial equation of the first degree where only integer solutions are sought. Its basic form is:

$$ax+by = c$$

where a, b, c are integers and $a, b \neq 0$.

Frobenius Coin Problem

The Frobenius coin problem states that for any relatively prime positive integers m,n, what is the greatest integer that cannot be written in the form am + bn or in other words the Frobenius number of m,n. Where gcd(m, n) = 1.

In a less formal way, the question is simply, what is the smallest amount of McNuggets that cannot be ordered from McDonalds using the standard sized boxes of 9 and 20 (m and n) the quantities that McDonald's originally sold chicken nuggets in.

So, what is the largest number that we cannot make in the form 9a+20b, this number is called the Frobenius Number.

Rem0	Rem1	Rem2	Rem3	Rem4	Rem5	Rem6	Rem7	Rem8
0	1	2	3	4	5	6	7	8
9	10	11	12	13	14	15	16	17
18	19	**20**	21	22	23	24	25	26
27	28	29	30	31	32	33	34	35
36	37	38	39	**40**	41	41	43	44
45	46	47	48	49	50	51	52	53
54	55	56	57	58	59	**60**	61	62
63	64	65	66	67	68	69	70	71
72	73	74	75	76	77	78	79	**80**
81	82	83	84	85	86	87	88	89
90	91	92	93	94	95	96	97	98
99	**100**	101	102	103	104	105	106	107
108	109	110	111	112	113	114	115	116
117	118	119	**120**	121	122	123	124	125
126	127	128	129	130	131	132	133	134
135	136	137	138	139	**140**	141	142	143
144	145	146	147	148	149	150	151	152
153	154	155	156	157	158	159	160	161
162	163	164	165	166	167	168	169	170
171	172	173	174	175	176	177	178	179

Above is a table of all the integers between 0 and 179. In bold are the multiples of 20 to represent the boxes of 20 Nuggets so these can be made. In addition to this all multiples of 9 can be made due to the boxes of 9 nuggets.

In the table below we take all the multiples of 20 and then continuously add nine to them. This is the equivalent of buying a box of 20 and then buying additional boxes of 9. This concludes with the highest value that cannot be made is 151 as all values that are higher than this can be expressed in the form 9a+20b.

Rem0	Rem1	Rem2	Rem3	Rem4	Rem5	Rem6	Rem7	Rem8
0	1	2	3	4	5	6	7	8
9	10	11	12	13	14	15	16	17
18	19	**20**	21	22	23	24	25	26
27	28	**29**	30	31	32	33	34	35
36	37	**38**	39	**40**	41	41	43	44
45	46	**47**	48	**49**	50	51	52	53
54	55	**56**	57	**58**	59	**60**	61	62
63	64	**65**	66	**67**	68	**69**	70	71
72	73	**74**	75	**76**	77	**78**	79	**80**
81	82	**83**	84	**85**	86	**87**	88	**89**
90	91	**92**	93	**94**	95	**96**	97	**98**
99	**100**	**101**	102	**103**	104	**105**	106	**107**
108	**109**	**110**	111	**112**	113	**114**	115	**116**
117	**118**	**119**	**120**	**121**	122	**123**	124	**125**
126	**127**	**128**	**129**	**130**	131	**132**	133	**134**
135	**136**	**137**	**138**	**139**	**140**	**141**	142	**143**
144	**145**	**146**	**147**	**148**	**149**	**150**	151	**152**
153	**154**	**155**	**156**	**157**	**158**	**159**	160	**161**
162	**163**	**164**	**165**	**166**	**167**	**168**	**169**	**170**
171	**172**	**173**	**174**	**175**	**176**	**177**	**178**	**179**

But we, as mathematicians, are intrinsically lazy people so we don't want to write these tables out every time we want to find the Frobenius number. Luckily for us, there is a formula

$$g(a, b) = ab-a-b$$

Where g(a, b) is an arithmetic operator similar to GCD or LCM which finds the Frobenius number. But how do we know this is true.

The Proof of the Frobenius Number Formula

The following logic provides a proof for Sylvester's Theorem (1884), which defines the Frobenius Number, g(a, b) = ab – a – b. This derivation is an adaptation of the standard modern algebraic proof, utilizing a contradiction to show that no non-negative integer solution exists for the Frobenius number.

Where *a represents how many boxes of x McNuggets we have. For example, if we had 4 boxes of 9 McNuggets a = 4 and x = 9 the same applies for b and y.*

We want to prove that g(a, b) cannot be written in the form ax+by where a, b, x, y ∈ N.
This means that the Frobenius number g (a, b) cannot be made from the sum of the multiples of our different boxes of McNuggets.

Assume:

$$ab-a-b = ax+by$$

We are taking this statement to be true, and will be proving it false later

$$\Rightarrow a(x+1) + b(y+1) = ab$$
$$\Rightarrow a \mid b(y+1)$$

The notation $n \mid m$ *means n divides m*

Since the gcd(a, b) = 1

This means that they are coprime and their only common factor is 1 therefore a cannot divide b so the part of the term b(y+1) that a divides must be in the y+1 component this is by Euclid's Lemma.

$$\Rightarrow a \mid (y+1)$$

Since y ≥ 0, then y+1 ≥ 1 since a is a positive integer that divides itself then it must be that a is at least y+1.

$$\Rightarrow y+1 \geq a$$

Similarly:

$$a(x+1) + b(y+1) = ab$$
$$\Rightarrow b \mid a(x+1)$$

As before b cannot divide a due to the definition of the problem

$$\Rightarrow b \mid (x+1)$$

Since $x \geq 0$, then $x+1 \geq 1$ since b is a positive integer that divides b then it must be that a is at least x+1.

So $(x+1) \geq b$.

By this manner we can show

$$y+1 \geq a$$
$$b(y+1) \geq ab$$

In addition:

$$(x+1) \geq b$$
$$a(x+1) \geq ab$$

Therefore, we can show that

$$a(x+1) + b(y+1) \geq ab+ab$$
$$a(x+1) + b(y+1) \geq 2ab$$

So by contradiction no such non-negative integers x,y exists to satisfy the equation.

This shows that the number from ab-a-b cannot be made from the boxes of McNuggets, but we are still yet to show that any number greater than ab – a- b can be made from our combination of McNuggets. Now let's prove that all integers greater than ab – a – b are possible.

$$T = \{ ax + by \text{ where } x,y \in N \}$$
We want to prove if $n > ab – a -b$ then $n \in T$

This initial line creates a set (a collection of numbers) called T which contains every possible value which can be made in the form ax+by.

If we can make a number with the same remainder when divided by a (n mod a) then we can make any number above it with the same remainder by simply adding a. Imagine a ladder and the first step is the first value n with that remainder the next step of the ladder can be made by adding a. Base step – 13 which 13 mod 12 = 1 so therefore any number above 13 when divided by 12 has remainder 1 can now be made.

$$R = \{0b,1b,2b,3b \ldots b(a-1)\}$$

This defines a set R which contains all the multiples of b which all have a different remainder of a as a and b are coprime. So, the largest number we need is b(a-1) luckily the Frobenius number *formula can help us out.*

$$ab – a – b$$
$$=b(a-1)-a$$

Therefore, we can show that the largest multiple of b we need is a above the Frobenius number therefore as each member of R has a different remainder each value above b(a-1)-a can be made. All we need to show now is that each number between ab-a-b and b(a-1) can be made and we have completed our proof.

The largest base step needed to cover all the remainders of mod a is b(a-1). Once this value is reached every subsequent number with that remainder can be formed by adding a. The largest impossible number is simply the step of a bellow the final base step.

$$b(a-1)- a = ab-a-b$$

Challenges in the Three-Variable Case

Finding the Frobenius number for more than 2 variables becomes increasingly more complicated. In this part we will discuss the problems that arise and some of the cases with three variables and why there no longer exists a general formula.

There is one case of three-variables in which the problem decomposes into the two-variable case. If one of the numbers is a factor of each other, for example 4 and 20. As we can make 20 for 4x5 we can just use the general formula for two-variables with the parameter as 4 and m (the other number) as 20 “lies on the 4 ladder”

The complexity really increases when we reach pairwise coprime (the GCD of every pair is 1). This is because the various ladders of each possible combination start to interfere with each other which makes it harder to identify which combination unlocks specific remainders.

So how do we actually work out g(a, b, c)? The most common way is to use Dijkstra’s Algorithm to find the shortest path between each remainder (nodes on the graph) is to find the first arrival of each remainder, then the Frobenius number is the *largest remainder minus a*. This is similar to how b(a-1) – a is the Frobenius number with two- variables.

Applications

The Frobenius Coin Problem surprisingly has a wide variety of applications. It can be used to determine the most efficient sizes of boxes of products like McNuggets to ensure almost any quantity can be ordered by customers. It can be used in mint offices to try and optimise coins and bank notes. It possesses uses in discrete mathematics as well when calculating the complexity of shell sort algorithms. Cryptography, the theorem can be used in the design of knapsack-style encryption, where security relies on the difficulty of finding specific combinations of numbers. While these examples highlight practical uses, the Frobenius problem extends far beyond packaging and coins, serving as a foundational concept in advanced fields like algebraic geometry, computational complexity, and the study of numerical semigroups.

Conclusion

What began as a relatively simple question in a McDonald's drive through regarding unpurchasable chicken nugget quantities ends with us expanding into a variety of different fields in mathematics and just skims the surface of number theory. We have seen the greatest common factor and Euclid's lemma appear providing elegant solutions for proving Sylvester's Theorem. We explored the problem in the elegant two variable case and dipped our toes into the inherently more complex cases with three-variables. The Frobenius Coin Problem is a perfect example of how an intriguing puzzle can have significant implications in diverse fields of mathematics.

Sources

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