From Toothpicks and 'Quirksets' to Fibonacci: The Miracle of Generating Functions

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1 Introduction

One of the key reasons I find mathematics so fascinating is because often by playing with toy problems *you* have created purely out of honest curiosity, you can stumble on a journey of self-learning new math. To be clear, if those same mathematical concepts were presented to you or me in a math class without sufficient motivation, we would surely not have felt the same thrill or relevance that one feels through those journeys. I highly suggest watching [1] for more context into my philosophy as to why and how to do math for fun.

This note presents one such mathematical toy problem I made up, the journey to solving which was quite satisfying and insightful to me.

2 The Problem and Motivation

A well-known result from elementary set theory states that a set with n elements has 2^n subsets. For example, the set `{A,B}` has $2^2 = 4$ subsets: `{}`, `{A}`, `{B}`, `{A,B}`.

To see why, consider building a subset element by element: for each element, you either choose to include it or not, giving you 2 choices. Since selections are independent, the multiplication principle gives $2 \cdot 2 \cdots 2$ (multiplied n times) $= 2^n$ subsets in total.

This is all good and simple. But notice how the argument hinges on the independence of choice. What if some quirky individual[1] imposed a restriction — say, counting only subsets containing **non-adjacent** elements from the original set? That is, if we include `B` from `{A,B,C}`, we cannot include any of its neighbors A or C. How many such subsets exist for a general n-element set?

[1] That's me

3 Preliminary Attempt and Recurrences

Okay. We just made ourselves a seemingly interesting problem[2]. Unlike before, the possible choices (to include or not include) for an element *depend* on whether one of its neighbors has been included.

Still, trying to morph the previous approach into something useful here, we can try assuming a choice of, say, 'include' for one element (like the first element) and see what "restrictions of choices" propagate through its neighbors and beyond, and then consider the case where we pick 'do not include' for the first element and see what happens. Then, we could maybe use the addition principle to add together the two counts. Sure, for a two-element set this works fine. But for three and above, we find ourselves having to assume choices for multiple elements, and keeping track of cases and sub-cases starts to cause a mild headache. It is messy:

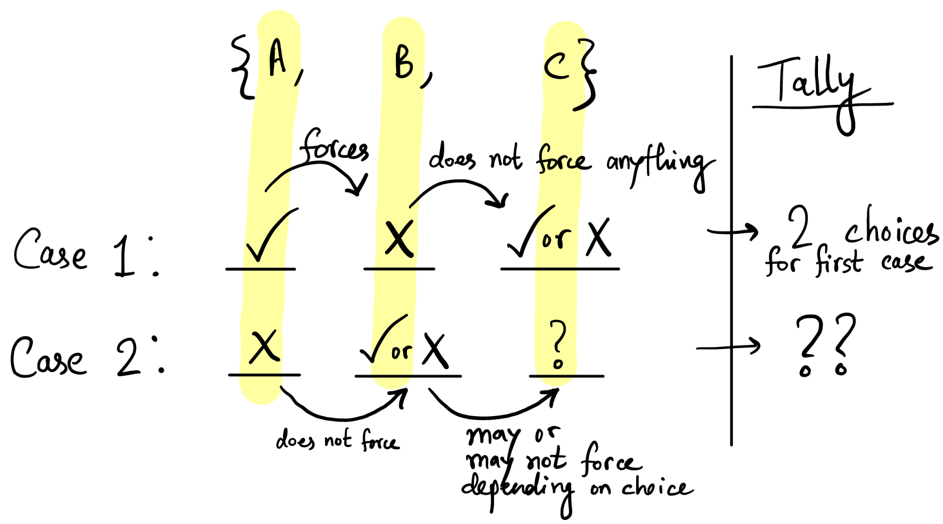

Figure 1: Messy

So, instead of feeling down, why not just play with the problem and find out some values by hand? For instance, there are 3 of our quirky subsets (why not call them **quirksets** from now on?) for $n = 2$. For $n = 3$, we can imagine a set `{A,B,C}` and get the quirksets `{}`, `{A}`, `{B}`, `{C}`, `{A,C}`. So, there are 5 quirksets for $n = 3$. Likewise, we can prepare the following table with some straightforward systematic ad-hoc counting:

[2]Spoiler: it is

n	$f(n)$
0	1
1	2
2	3
3	5
4	8
5	13
$\vdots$	$\vdots$

Here, we represented the number of quirksets for an n-element set with the function $f(n)$. If you are anything like me, you should sense something familiar here. Hmm, doesn't $2, 3, 5, 8, 13, ...$ look like ... part of... *THE FIBONACCI SERIES*!

Indeed, it seems like you can get any number on the right-hand column by adding together its two predecessors, which is characteristic of the Fibonacci sequence. Now, how do we actually know that this Fibonacci pattern holds forever? After all, many of us probably know about Moser's circle problem and watched [2]. However, for us, it turns out, it is actually quite easy to rigorously prove our recurrence conjecture using the same idea that failed us in Figure 1!:

Recurrence conjecture: $f(n) = f(n-1) + f(n-2)$ for $n \geq 2$

Proof. Considering $n \geq 2$, we can broadly classify our quirksets into two categories: ones containing the last element of the set and ones not containing it:

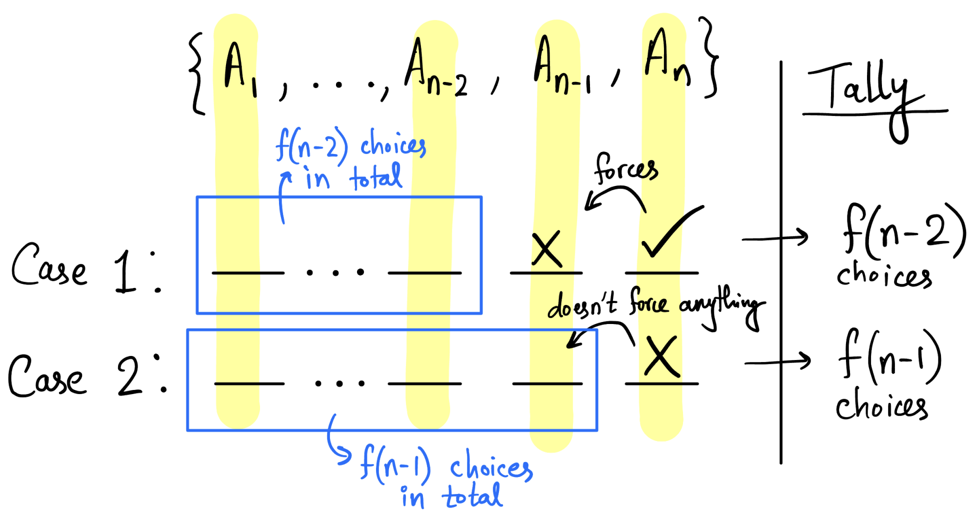

Figure 2: Not messy at all!

When we include the last element A_n, we can't pick its neighbor A_{n-1}, so we have to choose valid quirksets from the first $n-2$ elements, which can be done in $f(n-2)$ ways. Those quirksets, along with the last element A_n stuffed into each of them, form our final Case 1 quirksets. Very similarly, we get $f(n-1)$ number of Case 2 quirksets since the act of not including A_n forces nothing on the previous $n-1$ elements. Et voila! Using the addition principle, we can sum up our Tally column to get:

$$f(n) = f(n-1) + f(n-2)$$

□

If you ask me, a great lesson here is to not immediately throw away ideas that do not work; they may save the day later on.

Anyway, now that we have established the recurrence (which basically gives us the classic Fibonacci series shifted forward by two units), only one question remains: what IS the $f(n)$ for a given n. I mean, you *could* use the first two terms and the recurrence relation to calculate any term, but it would require A LOT of adding for large n. Like the previous pretty 2^n formula, can we get an explicit, closed-form formula for $f(n)$?

4 Toothpicks and Generating Functions: A Miracle

This is the point in my journey where I realized that I didn't know how to solve recurrence relations. Does an explicit nth-term formula even exist? Well, we will soon find out...

But not now. Why? I promise I will explain at the very end. Allow me to take a break from our narrative.

What I am about to discuss now might seem exceedingly irrelevant, but bear with me for a while. Did you know that, theoretically speaking, a single toothpick could contain all of the greatest works in history stored in a textual format? You name it: Hamlet, The Principia Mathematica, The Encyclopedia Britannica, and everything else.

Intrigued? Well, it's very simple actually (see Figure 3 and [3]). Take every letter from A to Z, and assign it a two digit number, say 00 to 25. Then the small letters could get the subsequent numbers, and then the punctuations — you get the idea. Then, take, say The Encyclopedia Britannica, and start reading it character by character, while scribbling the corresponding code on a piece of paper. Given you have enough patience, you will obtain a monstrous number by the end. Then, place 0 and a decimal point at the very front of the number. Congratulations, you have made yourself a number between 0 and 1. Now, take a $1cm$ toothpick, and simply measure out the number you have obtained (in cm) extremely accurately from one end of the toothpick and mark it using an extremely fine notch. Done!

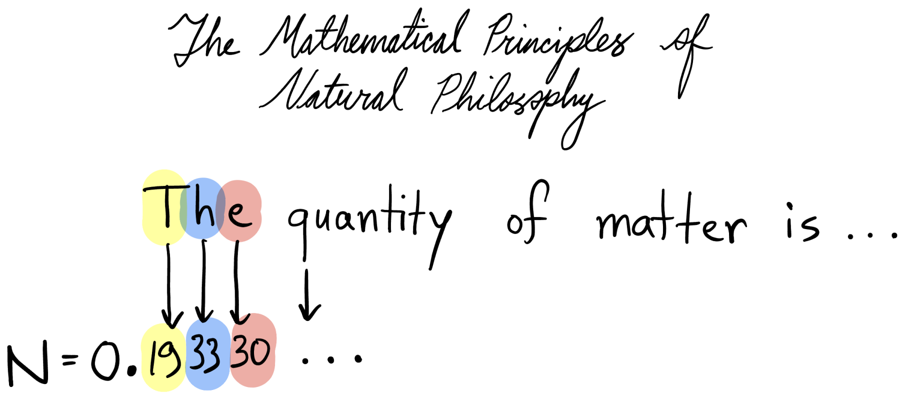

Figure 3: Example encoding of Newton's Principia

Anyone who needs to retrieve the gigantic text will simply[3] measure the distance of the notch from the end of the toothpick to an insanely high accuracy to get back your decimal number and can thus decrypt back the text.

Notice what happened there. As you might have guessed, the mathy bit lies in producing the decimal number. Suppose $c(n)$ denotes the two-digit code for the nth character of our gigantic text. Then the decimal number, N, that we obtain is:

$$N = \sum_{n=1}^{\infty} \frac{c(n)}{100^n} = \sum_{n=1}^{\infty} c(n) \cdot \left(\frac{1}{100}\right)^n$$

Why are we using powers of 100 in the denominator? Well, if we used 1 digit codes, then the nth digit would have a value of $c(n)$, but would be placed at the nth position after the decimal point, which would add a value of $\frac{c(n)}{10^n}$ to our final number N. Since we are using *two* digit numbers, we have to use powers of $10^2 = 100$ in the denominator.

Let's do something even more general. Let's forget about the specific $\left(\frac{1}{100}\right)$ in this case and replace it by x. Then,

$$N = \sum_{n=1}^{\infty} c(n) \cdot x^n$$

Why do this strange thing? Well, doesn't it excite you that this compact number N can contain ALL the information about the characters in a text? The number N probably doesn't have a mathematical significance since we are using arbitrary encoded

[3]Yeah, "simply". Just ask for an ruler with Planck's length-spaced marks. I don't know if that would be accurate enough though

characters, but what if we used something more mathematically structured instead? What if we used... OUR QUIRKY SEQUENCE $f(n)$?

All right, we are back to our narrative. First of all, let's define the number Q (for quirky obviously) as:

$$Q = \sum_{n=0}^{\infty} f(n) \cdot x^n = \mathbf{1} + \mathbf{2}x + \mathbf{3}x^2 + \mathbf{5}x^3 + \mathbf{8}x^4 + \cdots \tag{1}$$

Here, we started from $n = 0$ since that is first term of our sequence. Notice that this equation is actually a polynomial in x, which will make its manipulation familiar: it's just algebra[4]. First of all, let's think about what distinguishes Q from our earlier encyclopedia's decimal number. Well, unlike previously, each term of our sequence is linked with its two ancestors by $f(n) = f(n-1) + f(n-2)$. How might we try utilizing that? Here's something:

$$\begin{aligned}
Q &= \sum_{n=0}^{\infty} f(n) \cdot x^n \\
&= f(0) \cdot x^0 + f(1) \cdot x^1 + \sum_{n=2}^{\infty} f(n) \cdot x^n \\
&= 1 + 2x + \sum_{n=2}^{\infty} \big(f(n-1) + f(n-2)\big) \cdot x^n \\
&= 1 + 2x + \sum_{n=2}^{\infty} f(n-1) \cdot x^n + \sum_{n=2}^{\infty} f(n-2) \cdot x^n \\
&= 1 + 2x + \sum_{n-1=1}^{\infty} f(n-1) \cdot x^n + \sum_{n-2=0}^{\infty} f(n-2) \cdot x^n \\
&= 1 + 2x + \sum_{n-1=1}^{\infty} f(n-1) \cdot x \cdot x^{n-1} + \sum_{n-2=0}^{\infty} f(n-2) \cdot x^2 \cdot x^{n-2}
\end{aligned}$$

All we did was write down the first two terms explicitly and use the recurrence on the remaining sum. Notice that we changed the dummy variable from n to $n-1$ and $n-2$ for the two summations, respectively. This is because the summands *include* $n-1$ and $n-2$ in them as arguments of the function and exponents. Let's make the substitutions $a = n-1$ and $b = n-2$:

[4] Albeit with some prerequisite astuteness, as we will see shortly

$$
\begin{aligned}
Q &= 1 + 2x + \sum_{a=1}^{\infty} f(a) \cdot x \cdot x^a + \sum_{b=0}^{\infty} f(b) \cdot x^2 \cdot x^b \\
&= 1 + 2x + \left(x \sum_{a=1}^{\infty} f(a) \cdot x^a \right) + \left(x^2 \sum_{b=0}^{\infty} f(b) \cdot x^b \right) \\
&= 1 + 2x + x \cdot (2x + 3x^2 + 5x^3 + 8x^4 + \cdots) + x^2 \cdot (1 + 2x + 3x^2 + 5x^3 + 8x^4 + \cdots)
\end{aligned}
$$

If whatever we did up to this felt bizarre, that should definitely change now. A quick inspection shows us that the sums in the brackets are suspiciously close to the form of Q in equation (1). In fact, the expression in the first bracket is just $Q - 1$ and the second is simply Q itself!:

$$
Q = 1 + 2x + x(Q - 1) + x^2(Q)
$$

$$
\implies Q = \frac{1 + x}{1 - x - x^2} \tag{2}
$$

Take a deep breath, because we have come a long way. We have found an explicit formula for the *sum* of the increasing powers of x, where the coefficients are our $f(n)$. If you plug in a concrete value[5] for x, say $x = 1/2$, you get a concrete value for Q, which I find fascinating since we are able to evaluate something infinitely intricate which *embeds* our sequence $f(n)$.

But, do not forget what our original intention was. We want an explicit formula for $f(n)$, the *coefficients* of powers of x in our sum. Is it possible to get information about *individual* terms from a sum? Think about it.

Well, here's a snippet from the early pages of my math notebook proving the formula for the sum of an infinite geometric series:

[5]But, what if you plug in $x = 1$? Can Q be negative? What guarantees that the sum Q converges? Unfortunately, this essay would be too long if we start exploring tangents... But do feel free to explore – that's what math is all about!

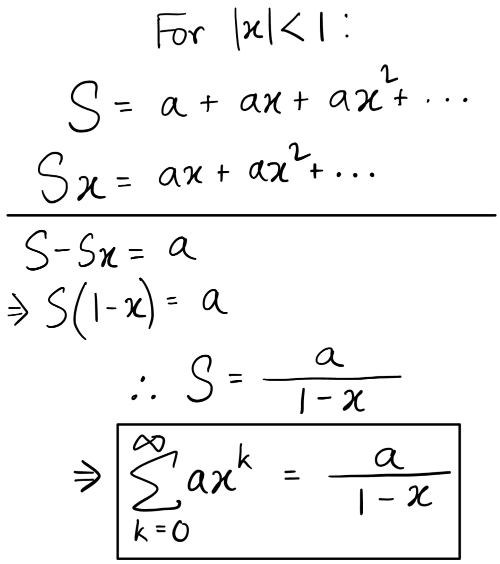

Figure 4: Snippet from notebook

A curious thing here is that a *fraction* of the form $\frac{a}{1-x}$ gives us a power series expansion. Look at equation (2); we have a fraction. Could we try obtaining a power series from *that*?

Well, our denominator is quadratic. But fear not! Using the powerful algebraic tool of partial fraction decomposition, we can make a way forward by turning our fraction into two fractions with linear denominators:

$$\begin{aligned} Q &= \frac{1+x}{1-x-x^2} \\ \implies -Q &= \frac{1+x}{x^2+x-1} \\ \implies -Q &= \frac{1+x}{(x-\alpha)(x-\beta)} \end{aligned}$$

The roots were obtained by using the quadratic formula on the denominator; $\alpha = \frac{-1+\sqrt{5}}{2}$ and $\beta = \frac{-1-\sqrt{5}}{2}$. Anyway[6]:

$$\begin{aligned} -Q &= \frac{1+x}{(x-\alpha)(x-\beta)} \\ \implies -Q &= \frac{A}{(x-\alpha)} + \frac{B}{(x-\beta)} \end{aligned} \tag{3}$$

[6]This is probably the most nonchalant word to say when the golden ratio popped up in the previous sentence

Multiplying the two equivalent forms above by $(x-\alpha)(x-\beta)$ gives $1+x = A(x-\beta) + B(x-\alpha)$, which is true for ALL x. Plugging in two random[7] values for x gives two equations for A and B:

$$\begin{aligned} x=\alpha \ gives: &\quad (1+\alpha) = A(\alpha-\beta) \implies A = \frac{1+\alpha}{\alpha-\beta} \\ x=\beta \ gives: &\quad (1+\beta) = B(\beta-\alpha) \implies B = \frac{1+\beta}{\beta-\alpha} \end{aligned}$$

Now that we know what α, β, A, B are, there's one last thing left to be done: turn equation (3) into the form from our math notebook in Figure 4:

$$\begin{aligned} -Q &= \frac{A/\alpha}{(x/\alpha)-1} + \frac{B/\beta}{(x/\beta)-1} \\ \implies Q &= \frac{A/\alpha}{1-(x/\alpha)} + \frac{B/\beta}{1-(x/\beta)} \end{aligned}$$

Ta da! Both fractions are now of the form $\frac{a}{1-x}$. So we can use Figure 4 to write:

$$\begin{aligned} Q &= \sum_{n=0}^{\infty} \frac{A}{\alpha} \cdot \left(\frac{x}{\alpha}\right)^n + \sum_{n=0}^{\infty} \frac{B}{\beta} \cdot \left(\frac{x}{\beta}\right)^n \\ &= \sum_{n=0}^{\infty} \left(\frac{A}{\alpha^{n+1}}\right) x^n + \sum_{n=0}^{\infty} \left(\frac{B}{\beta^{n+1}}\right) x^n \\ \therefore Q &= \sum_{n=0}^{\infty} \left(\frac{A}{\alpha^{n+1}} + \frac{B}{\beta^{n+1}}\right) x^n \end{aligned} \tag{4}$$

Yes, if it clicked to you, we can start celebrating. Basically, comparing equation (4) with the very first equation (1), we see that by trying to convert the fraction (2) into a power series, we essentially unveiled the identity of the coefficients in front of the powers of x; which is our sequence $f(n)$![8]. Thus, finally, we have:

$$f(n) = \frac{A}{\alpha^{n+1}} + \frac{B}{\beta^{n+1}},$$

where the constants A, B, α, β have been defined previously. We can even empirically verify that this formula works for the first few terms using Desmos:

[7] Okay, maybe not random. Algebraic astuteness is relevant here

[8] Don't confuse exclamations with factorials, nerd

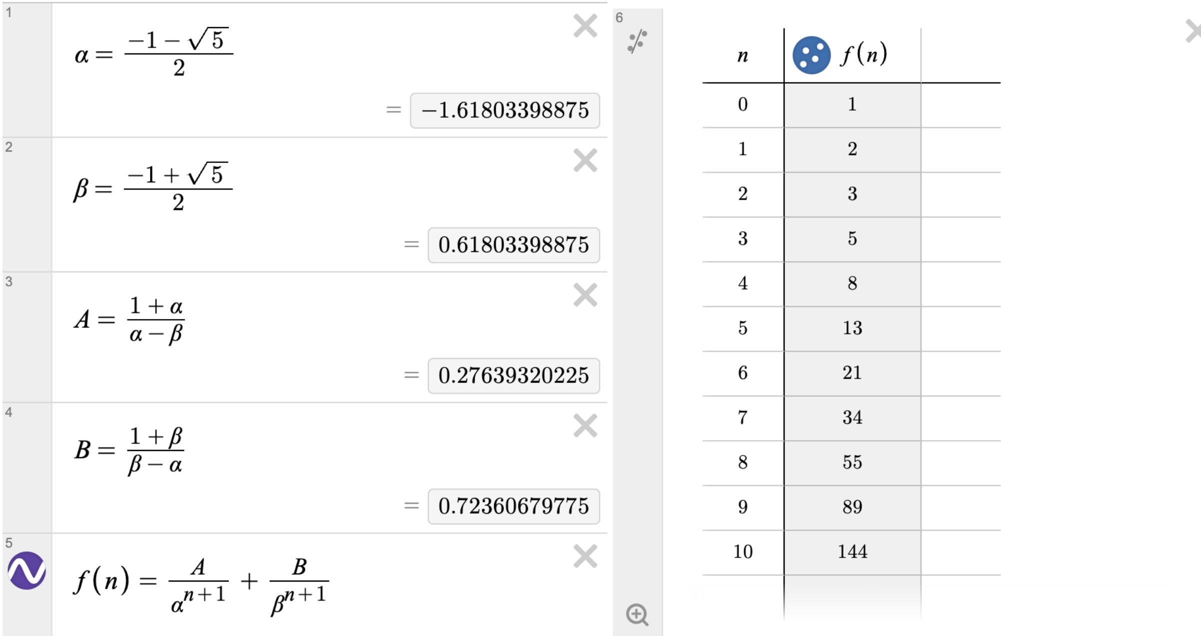

Figure 5: Empirical verification

Reflecting on this miraculous resolution, the star of our show, variable Q, is known as the generating function of our sequence. Observe that *unlike* regular series, we did not have to worry about the actual variable x or whether Q converges or not. For this reason, a generating "function" is not considered to be a function in the sense that it doesn't have a domain and co-domain; rather, it is its algebraic *structure* that we utilize.

5 Conclusion and Farewell

This has been quite a journey, and an enjoyable experience for me to narrate it as well. As promised, let me explain the bizarre "toothpick" break in the narrative that occurred a few pages back. In short, when I first learnt about generating functions to solve our quirkset problem, it felt more or less like an unmotivated magic. Years later, while watching [3], I got the germ of the idea to motivate this magical concept.

Given the scope of the essay[9], many of the tangential explorations had to be restricted. For instance, the quadratic we had in one of our denominators with roots α and β is known as the *characteristic polynomial* of our recurrence relation. Characteristic polynomials show up plentifully in math, from differential equations to linear algebra, and demand an exploration in their own right.

And to me, that *is* the essence of math. Sailing on your own voyages through an intricately intertwined world of truth. I hope you set your next sail soon!

[9]And the fact that I am dangerously approaching the word limit

References

[1] The Mosaic Problem - How and Why to do Math for Fun,
https://youtu.be/D3dp5RBmPcs?si=fymXTHN4mDvxBn0w.

[2] How They Fool Ya (live) — Math parody of Hallelujah,
https://youtu.be/NOCsdhzo6Jg?si=DZi51LJyoq0oba7x.

[3] This Toothpick Contains Everything Ever Said (Infinity Part 3),
https://youtu.be/wXHUn376IMo?si=kwJwrOnEFI4NR9lr&t=56.

[4] Paul Zeitz, *The Art and Craft of Problem Solving*, John Wiley & Sons, 2017.