Minimising The Moves

How the Rubik's cube World record has broken the 3 seconds barrier?

By Leo Laifer, UK Year 12

1 – The art of the cube

The Rubik's cube, first created by Erno Rubik in 1974 is a simple premise:

- A solid cube made up of 6 different coloured sides
- 26 smaller cubes connected to a rigid six-sided core
- These smaller pieces can be broken down into smaller categories:

6 centre pieces

12 edge pieces

8 corner pieces

Whilst the cube is deceptively easy to understand, the possibilities of colour combinations are essentially endless. Some say it is impossible to solve as when you get one part solved you mess it up solving another part (please keep that in your head). Which begs the question, how can someone take a puzzle with almost an infinite number of permutations and return it to its solved state? Let alone in less than 3 seconds?

1.1 How many permutations are there?

Given the standard 3x3 cube with the parameters discussed prior:

$$Permutations\ of\ corner\ pieces = 8! \times 3^7$$

This product is determined by the 8 corner pieces, hence 8!, multiplied by the fact each corner piece can be in 3 different states (see visualisation below). However as one of the corner pieces is dependent on the state of the others, we know 3 raised to the eight would be one too many states of the corner. Hence 3 raised to the seven is the number of states.

$$Permutation\ of\ edge\ pieces = 12! \times 2^{11}$$

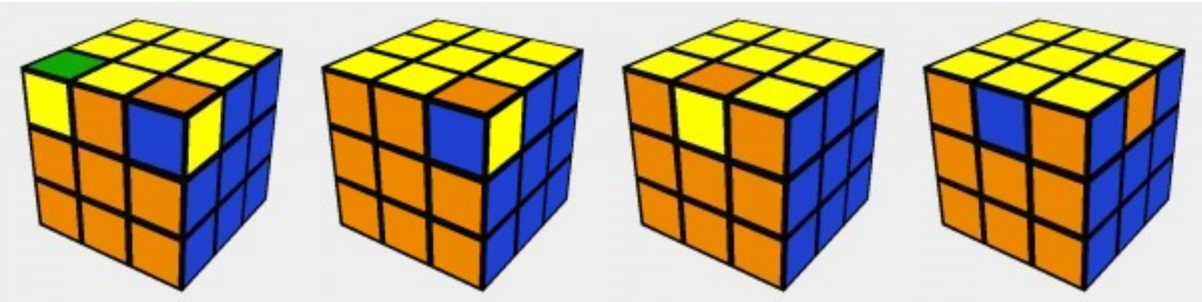
Similarly, the 12 corner pieces each have 2 states they can be in (see visualisation below). Moreover, one edge is dependent on the other 12 due to edge parity hence 3 raised to the 11 is the number of states.

$$Permutation\ of\ Rubiks\ Cube = (12! \times 2^{11}) \times (8! \times 3^{7}) \times \frac{1}{2}$$

= 43,252,003,274,489,856,000 possible arrangements

The half arises from the idea that two pieces cannot swap independent of other pieces (see visualisation below) hence only 50% of the combinations would exist, hence the product is multiplied by 0.5.

Visualisation



This shows the nature of cubes which have parity – as these cases are impossible to have without taking apart the cube they are subtracted when working out the total combinations.

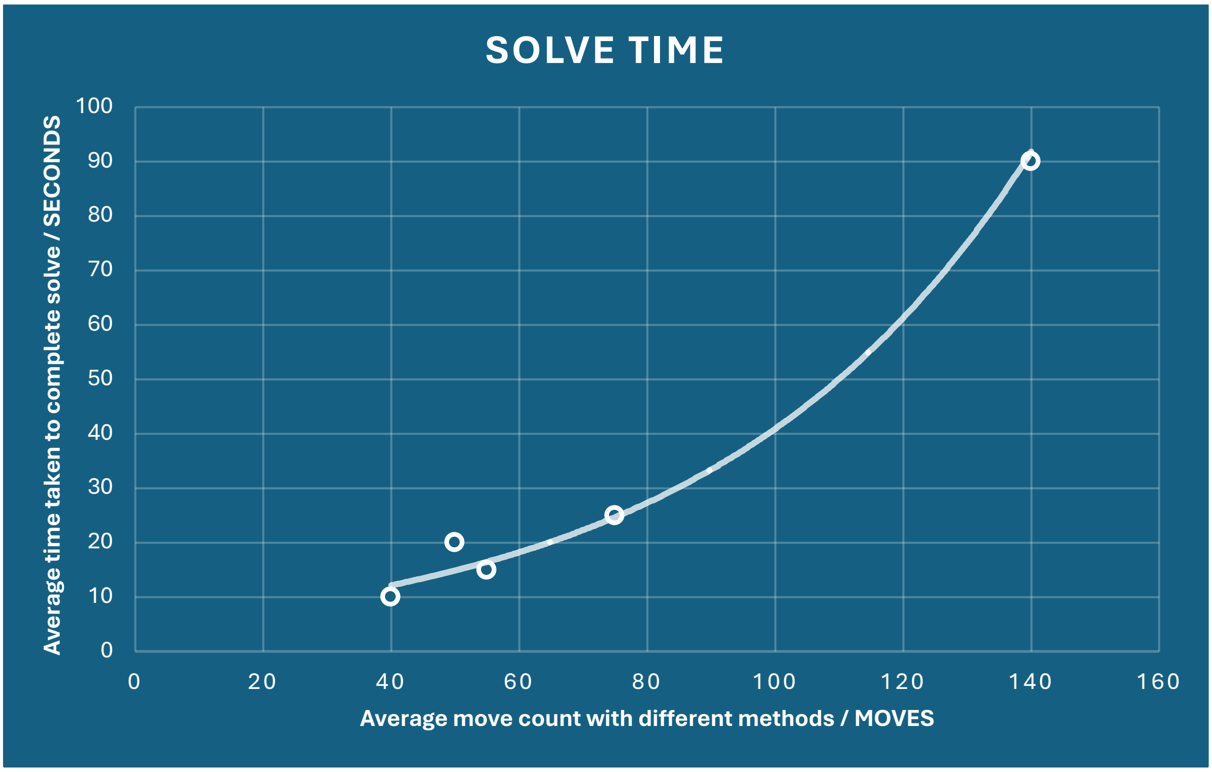
1.2 Methodology to solving

Swiftly after the initial arrogance of ‘a simple puzzle’ (we have all been there) one may turn to a method to solve the cube:

- CFOP (Cross, F2L, OLL, PLL)
  Average solve is 10-20 seconds
- Roux (Method of building blocks around the cube to solve)
  Average solve is 40-50 seconds
- ZZ (involves orienting al the edges then solving the corners)
  Average solve is 20-30 seconds
- ZBLL (Advanced method of extremely fast solving)
- *Average solve is 5-15 seconds*
- Beginners’ method (solve each layer slowly using multiple algorithms)
  Average solve is 1.5-2 minutes

Whilst this paper does not discuss in detail how to solve the cube – comparing methods gives us some key factors to consider when investigating the question.

Method	Number of Algorithms required to learn	Average move count of the solve (hence the name of the paper)	Time taken to learn the method
Beginners' method	4-8	130-145 moves	Approx. 2 weeks to 1 month
CFOP (me!)	78-90	50-60 moves	Approx. 6 months to 1 year
Roux	42-60	50-55 moves	Approx. 6 months to 1 year
ZZ	69-84	50-60 moves	Approx. 6 months to 1 year
ZBLL	472-493!!!	30-50 moves	Approx. 3 years to 9 years



From this graph, it would be fair to deduce that as you increase in the number of moves you perform in a solve it will rapidly increase your overall time taken, This may be because for every move there is the physical movement of the cube and there is also a thinking time where your brain needs to decide on the algorithm or steps next needed – whereas on the faster side there is less thinking time as learning more algorithms had led to more cases not needing to be thought about.

2 – Accelerating the algorithms

2.1 What makes a good algorithm?

To speed up the world record solving time, the most optimal algorithms must be used.

In terms of optimization, we cannot just think about move count as some short algorithms are less ergonomic to perform than their slightly longer counterparts – hence we can come up with a relationship of the following:

$$Time = (move\ count)(time\ per\ move) + (thinking\ time)$$

Hence if you have a more ergonomic algorithm both the time per move and the thinking time will decrease greater than the effect of the move count maybe slightly increasing.

A good algorithm also needs to physically be easy to perform i.e. always allowing a tight grip on the cube which will prevent incidence such as a cube flying out your hands mid solve – and this has happened before!! And this is not very good for fast solving as you may expect!

Lastly, a useful algorithm for speed solving will have an element of **lookahead** which is a term used in speed solving to describe the process of visualising the movement of pieces before it happens to attempt to think about the next step and reduce thinking time. The person who is most known for their lookahead is Max Park who is once said to be able to visualise a whole solve just by imagining the turns and using efficient algorithms. Algorithms with good lookahead will contain 'blocks' which are pieces of solved cube that are together which make it easier to imagine the state of the cube when an algorithm is done.

2.2 How do we make algorithms?

Algorithms are primarily made using a pre-determined set of commutators. Commutators are fundamental small set of moves which specifically move individual pieces to a desired position on the cube.

Commutators in Rubik's typically follow this form (and its inverse) to have a minimal effect on the rest of the cube (so they don't affect other pieces):

$$typical\ form\ of\ commutator: [a, b] = aba^{-1}b^{-1}$$

Which essentially means "do move A, then move B, then undo move A, then undo move B"

$$inverse\ form\ of\ commutator: [a, b]^{-1} = bab^{-1}a^{-1}$$

Which essentially means "do move B, then move A, then undo move B, then undo move A"

We can also use the mathematical property that the commutator $[a, b]$ followed by $[a, b]^{-1}$ results in **no change to the cube** which we can use to perform 2 commutators and then the inverse of the first to change a few pieces on the cube without affecting the rest – remember I said to keep this idea in you head.

The next section contains Rubik's cube notation; hence you may want to understand how it works as it is how cubers are able to discuss and create algorithms.

The most common triggers are known as 'Righty and Lefty Algorithms' (they also are sometimes called the sexy moves!) and they are used to help us think of all the common commutators that we can simply piece together to form algorithms:

Righty algorithm: R U R' U' (moves the 2-bottom front right vertical pieces into the top layer)

Lefty algorithm: L' U' L U (moves the 2-bottom front left vertical pieces into the top layer)

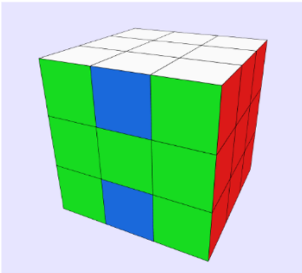
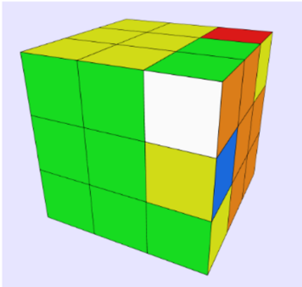
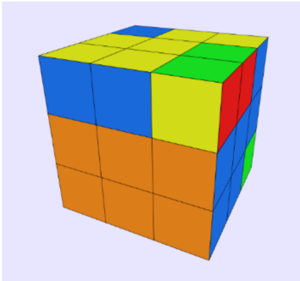
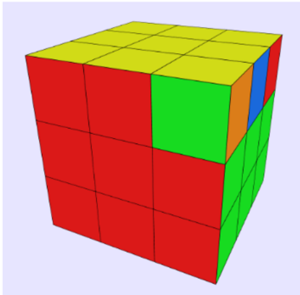
See how the two most common algorithms follow the standard commutator and inverse commutator form. That is real life mathematical modelling!!

Other commutators include:

D L' U L D' (moves top from edge into the left side edge – without affecting any bottom corners)

M2 U2 M2 U2 (swaps opposing front facing edges on top and bottom.

Visualisation

We can use these fundamental commutators and piece them together to get longer algorithms that are useful to solve the cube. Using what we know, we are going to create an algorithm used to make a real-world record in 2008. As an exercise you can attempt to find some more commutators and play around with their interactions – or try find this formula.

On the last step where we permute the top layer of the cube, we need to change the position of the top pieces without changing their orientation (sounds like the perfect time for some Righty algorithms). We can fix together different commutators (in the brackets) to get us our required result:

(R U R’ U’) R’ F (R2 U’ R’ U’) (R U R’ F’)

And that is an algorithm! The name of this one is a T-Perm, because the edges and the corners flip in a T shape!

The use of commutators is also very good for the memorisation of lots of algorithms – remember those 80 or so formulas. This is because your body can refine the commutators/triggers to be fast and that allows for algorithms to just glue together the commutators into a bigger sequence making muscle memory more effective if you only practise the core commutators.

2.3 Using these algorithms

In conclusion over these algorithms, we can use triggers to effectively speed up the time taken per move by mastering the muscle memory of simple sequences. In terms of getting to the world record times, commutators are of great use to generate new algorithms that people can learn which are extremely case specific- hence you wouldn’t learn if you were a beginner. Using these commutators, we can piece together essentially ANY algorithm we want to minimise your moves (hence the name!) and decrease thinking time which were are two key factors when considering decrease in total time taken.

3 – Manufacturing luck

- Teodore Zajder (2.76s)
- Yiheng Wang (3.08s)
- Max Park (3.13s)
- Feliks Zemdegs (4.22s)

Here is a list of some of the world record 3x3 holders with Teodore Zajder being the fastest right now (9 years old btw!). One thing in common with the majority of these records is there was blatant luck at some point in their solve – usually in the form of a last layer skip/ PLL skip.

The term Last layer skip refers to the idea of solving the first two layers of the cube and unintentionally then having the top layer solved already for you.

It would be inefficient to attempt to 'bank' on a last layer skip as the probability is so low. Therefore, cubers have methods with their algorithms to attempt to limit or get rid of any solving required on the last step.

How rare is a last layer skip?

Once the first two layers are solved there is 4 edges that still need to be in the right spot, so:

$$P(edge\ placement) = \frac{1}{4} \times \frac{1}{3} \times \frac{1}{2} \times 1 = \frac{1}{4!}$$

To orient the edges correctly there is a 1 in 2 chance it is flipped in the right way (we subtract one from the index for the same reason as in 1.1):

$$P(edge\ orientation) = \left(\frac{1}{2}\right)^3$$

The corners need to be in the correct placement, and there are 4 left as well:

$$P(corner\ placement) = \frac{1}{4} \times \frac{1}{3} \times \frac{1}{2} \times 1 = \frac{1}{4!}$$

Lastly the corners have three ways of being oriented and there are four of them so:

$$P(corner\ orientation) = \left(\frac{1}{3}\right)^3$$

We would need to half this total probability as you can't have only two swapped pieces (parity):

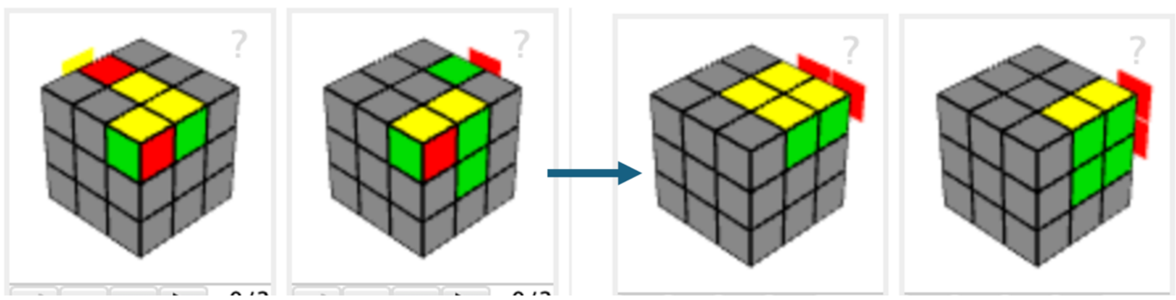
$$P(last\ layer\ skip) = \frac{1}{4!} \times \frac{1}{4!} \times \left(\frac{1}{3}\right)^3 \times \left(\frac{1}{2}\right)^3 \times \frac{1}{2} = \frac{1}{248{,}832}$$

Hence, I think it seems reasonable for these competitive cubers to not want to pray for these odds.

Techniques of block building

Cubers will use predictable methods of solving to reduce the amount of work on the last step. Such as 'block building' which is a method of grouping solved pieces which increases the probability of having good orientation and permutation of pieces by the end of the solve. Since these high level performers do so many solves, by decreasing the probability of a skip and increasing the sample size of the amount of solves that they do, a solve with a skip is deemed as more probable which allow them to make riskier plays within the solve (such as harder algorithms) as the likelihood of the payoff would be much greater i.e. no more moves required to do with a skip.

Some block building is done earlier on in the solve which restricts your choices as you don't want to mess up the blocks. Examples of these are:



Which shows how similar pieces can be stored together which increases the probability of having a last layer skip by already having some of the pieces in the correct orientation, lowering the factor of $\left(\frac{1}{2}\right)^3$ to $\left(\frac{1}{2}\right)^2$ or even lower, for example.

4 – Wrapping up the solve

The metamorphosis of the Rubik's cube from a humble toy to an intricate and competitive sport is a story that resonated with me well. We see how clever maths is used to accelerate how people can solve it in ways that were never thought possible before. Rubik's cubes offer a perfect example of real-life mathematical modelling that can shape people's lives – creating events, communities and friendships.

Thank you very much for reading and as an exercise to the reader – please go learn how to solve a Rubik's cube – it will fascinate you and bring you at least 30 minutes of entertainment. ☺

Sources:

ALL VISUALISATIONS WERE GENERATED BY ME USING

https://virtual-cube.net

https://lar5.com/cube/blox.html

INFO ON WORLD RECORDS

https://www.speedsolving.com/wiki/index.php?title=History_of_World_Records/3x3x3

INFO ON SOLVING METHODS

https://www.youtube.com/@JPerm