Money’s worst fear: Can stock crashes be modelled via stochastic calculations?

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Introduction

Picture this.

A mansion in the green fields of Switzerland, overlooking a farm. The sun is shining down on your home. And at the same time, you see the money rolling in.

This is a reality for many.

And why? Because of one word.

Stocks.

Or as Matthew Macghonaeughy says Shtocks.

It sounds easy, doesn't it?

Put money into a company, get more money back. Let me spoil it, it's not that easy.

In fact, events like what happened in 2008 are the reason why.

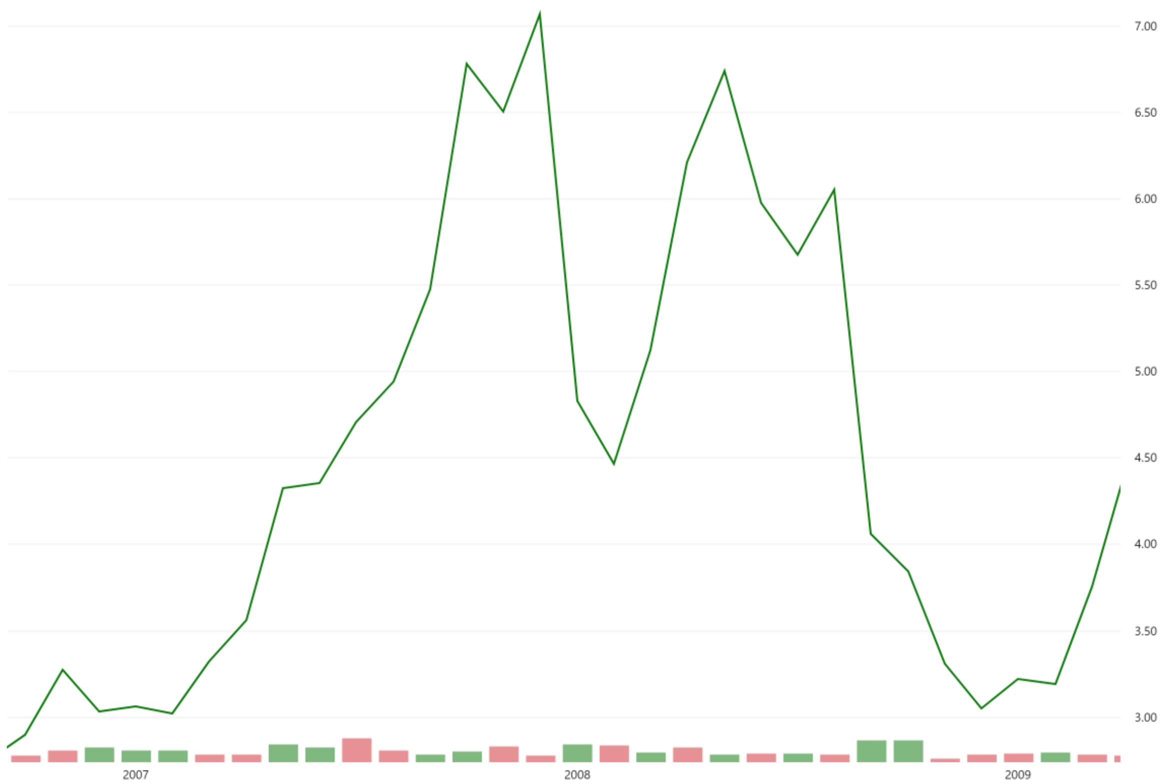
Defining the Problem

'Money money money, must be funny...' Oh sorry, I was singing a money song. Lets begin.

A stock crash is defined as a quick and dramatic drop in the price of a stock, which rapidly reduces the worth of a stock, reducing paper wealth.

I will be focusing on one stock in particular, Apple Inc or AAPL.

Not the fruit, but the massive tech company.



The graph above represents the crash of Apple between the peak of 2008 and the low of 2009. This dip may not seem like much, but mathematics reveals a dip of 43%.

The model will capture:

- Normal Price Evolution (diffusion process)
- Market Wide Systemic Shocks (crisis contagion)
- Sudden crash jumps (panic selling)
- Changing Volatility and Liquidity

Empirical Characteristics of Apple's Crash

1. Large drawback of the stock of 43% dip from January 2008 to January 2009.
2. Contagion and Market Correlation; during market crashes, stock volatility occurs simultaneously rather than separately.
3. Jump Behavior involves events such as Lehman Brothers collapsing leading to abrupt stock crashes rather than gradual declines, as the diagram shows.

Begin the General Geometric Brownian Equation (GBM) to model stock price

- $dS_t = \mu S_t dt + \sigma S_t dW_t$

Variable	Definition
S_t	Stock price when time = t
μ	Expected return
σ	Volatility
dW_t	Wiener Process, Yes quite funny. (Random Shock)

- Basic model representing how prices of a stock can be predicted and shown
- Returns are normally distributed
- Prices are continuous
- Variance grows with time
- However, the model fails to account for
    - Jumping Prices
    - High Volatility
    - Systemic risks affecting stocks

We can then convert this into a multiplicative form

- $dS_t/S_t = \mu dt + \sigma dW_t$
- We are factoring out the Stock Price variable, therefore dividing the differentiated value of Stock Price value by the Stock price to receive the expected return.
- Representing how returns are proportional to price
- This is as dS_t/S_t is simply the change in value of stock price divided by the original price, representing the expected return.

Introduction of Crisis Contagion

I love this part! A reality check.

Crisis Contagion is the process of which an external factor is to be considered to find the impact of it onto a stock.

During the 2008 financial crash, stock prices, especially tech stocks as large as Apple, were affected by widespread systemic shocks.

- Banking failures such as those of the Lehman Brothers
- Liquidity issues
- Credit market collapsing

This can be represented by a systemic stress variable

$$C_t$$

This variable can represent events such as

- Volatility index
- Market liquidity indicators and their issues

This systemic stress variable reduces the expected return rate that can be modelled via this formula

$$\mu_{effective} = \mu - \beta C_t$$

The constant β represents crisis sensitivity. This reveals the degree of responsiveness a stock is with respect to a change in asset price, volatility issue or an extreme market event, acting as a multiplier to determine whether an asset is highly susceptible to price issues or changes.

It is shown via this model as μ represents the expected return rate. The effective return rate is shown via the constant of crisis sensitivity multiplied by the crisis contagion. This is because the constant is the same throughout representing the volatility of the stock itself, and it is multiplied by the Crisis Contagion variable that is determined by the events occurring such as the Lehman Brother Collapse, representing the damage of return.

Therefore, the theoretical expected return in normal conditions minus the (constant multiplied by the event volatility variable) represents the effective return rate.

This can then be substituted into the multiplicative form equation

$$dS_t/S_t = (\mu - \beta C_t)dt + \sigma dW_t$$

The original expected return can be replaced by the effective expected return rate.

This model now includes macroeconomic influences and contagions.

Good thing this part didn't turn into a crisis, get it?

Add Jump Processes

Don't worry, no need for you to jump.

A Jump process tries to account for unsmooth crashes.

- Bank failures
- Panic Sell
- Large liquidation processes

These can be modelled using Poisson Processes

Since Poisson processes and Jump Motions σdW_t account for small market fluctuations which have 1 to 2 percent price changes. GBM assumes continuous price paths and infinitely small fluctuations.

Many stocks during the 2008 crash moved by 30%+, like Apple, which GBM doesn't account for.

This rapid jump and process can be defined by dN_t

The Poisson Process models the random arrival of events

Variable	Definition
N_t	Number of Crash Events up to time = t
dN_t	The rate of crash events with respect to time
λ	Average Crash Frequency

This process satisfies

$$P(N_{t+dt} - N_t = 1) = \lambda dt$$

$$P(N_{t+dt} - N_t = 0) = 1 - \lambda dt$$

Thus:

Most of the time $dN_t = 0$ as rapid changes in price rarely happen.

However, occasionally when the stock market is faced with problems, the probability spikes up, resulting in a higher chance for $dN_t = 1$, capturing rare shock events.

λ is the jump intensity

We then move onto defining the jump size

Defined by the variable of J_t

If a jump does occur, the model can be defined as the transfer of S_t to $S_t e^{J_t}$

Thus $J_t = ln(S_{after} / S_{before})$

The effects can be deduced to:

$J_t dN_t$

For example, if dN_t is equal to 1, then $J_t dN_t$ is equal to J_t, meaning the price rises instantly by the multiplier, signifying the multiplicative effect of a crash.

This can be inputted into the overall model to receive:

$$dS_t / S_t = (\mu - \beta C_t)dt + \sigma dW_t + J_t dN_t$$

Known as the **MERTON JUMP DIFFUSION MODEL**

The variable $J_t dN_t$ represents the multiplier effect of a crash with respect to the change in the number of crashes.

<u>Consider the Drift</u>

We begin with the Merton Jump Diffusion Model.

We take the expectation as $E[dW_t] = 0$

This is because, during market crashes, the effect of small market jumps and effects are almost miniscule and negligible due to the main crash event taking precedence and slight market movements are excludable. Causing the diffusion term to disappear. Therefore, the expected return becomes:

$$E[dS_t / S_t] = (\mu - \beta C_t)dt + E[J_t dN_t]$$

We can transfer the property of the jump variable from $E[J_t dN_t]$ to $E[J_t]\lambda dt$

Thus: $E[dS_t / S_t] = (\mu - \beta C_t)dt + \lambda E[J_t]\ dt$

This changes our expected return as initially it was supposed to be $\mu - \beta C_t$

To correct this, we subtract the expected jump contribution.

This can be done via introducing variable k.

Where $k = E[e^{J_t} - 1]$ which comes from the multiplicative nature of stock values, showing us the effect of a cumulative crash.

When a jump does occur, the stock price changes to S_t becomes $S_t e^{J_t}$

So, the proportional price change is e^{Jt} - 1

Therefore, $k = E[e^{Jt} - 1]$

Because previously we identified jumps occurring at a rate of λ, the jump contribution per unit time becomes λk

We subtract this from the initial expected return rate, giving us a new expected return of

$$\mu - \beta C_t - \lambda k$$

This can then be substituted into the formula, becoming:

$$dS_t / S_t = (\mu - \beta C_t - \lambda k)dt + \sigma dW_t + J_t dN_t$$

This is fascinating as we assume an expectation for the micro drifts of each stock on their trajectory, creating a more psychological aspect to the model.

Account for changing volatility

Since we are modelling for the 2008 financial crash, we must consider key volatility clustering of the stock market. This is when a high volatility period is followed by another period of high volatility, causing these periods and the effects of these to cluster.

During the 2008 crash, not only was there a period of high volatility but also it had effects such as high government debt burden, banking sector crisis and a huge global recession.

We can model this via the use of the Heston process.

The Heston process is a model which utilizes volatility, variance and time variables to come with an outcome of the volatility of the stock, affecting the variable σ causing it to be affected by the value of time resulting in σ_t.

This is the result of the Heston model:

$$d\sigma^2_t = K(\theta - \sigma^2_t)dt + \xi \sigma_t dZ_t$$

Variable	Definition
K	Mean reversion speed
θ	Long run Variance
ξ	Volatility of volatility

dZ_t	Secondary Brownian Motion

1. The mean reversion speed is the average time it takes a stock to reabsorb the shock it experienced previously. In this case, we are using Apple as a stock to determine how long it takes the stock to reabsorb the damage it faced during the 2008 crash, and its recovery time.
2. The Long run Variance is the measurement of the deviation of the stock from its expected trajectory, for example, the stocks volatility can be measured this way as the LRV measures how much the stock shifts from its path.
3. The volatility of volatility (VVIX) is the measure of the cluster we discussed earlier, revealing the overall effect the stock faces over the period, causing secondary effects.
4. The secondary Brownian motion is being used to determine the normal variable component within the Heston variance model as it allows us to see the effects of the volatility.

This leads the model to be developed into:

$$dS_t/S_t = (\mu - \beta C_t - \lambda k)dt + \sigma_t\, dW_t + J_t dN_t$$

<u>Replicate crash probability based on market crises level</u>

As we are trying to replicate the crash probability of the 2008 crash, we must consider variables that are at play during times of financial stress.

This will increase jump intensity causing our variable of λk to be more dynamic, shown by the model:

$$\lambda_t = \lambda_0 + \gamma C_t$$

λ_0 = The baseline crash probability

γ = Contagion strength

These variables reflect the initial crash chance of stock due to the financial pressure, and how likely it is to spread and circulate around the economy affecting the same stock again, respectively.

<u>Re-substitution of all variables</u>

Finally, we substitute all the variables back into form:

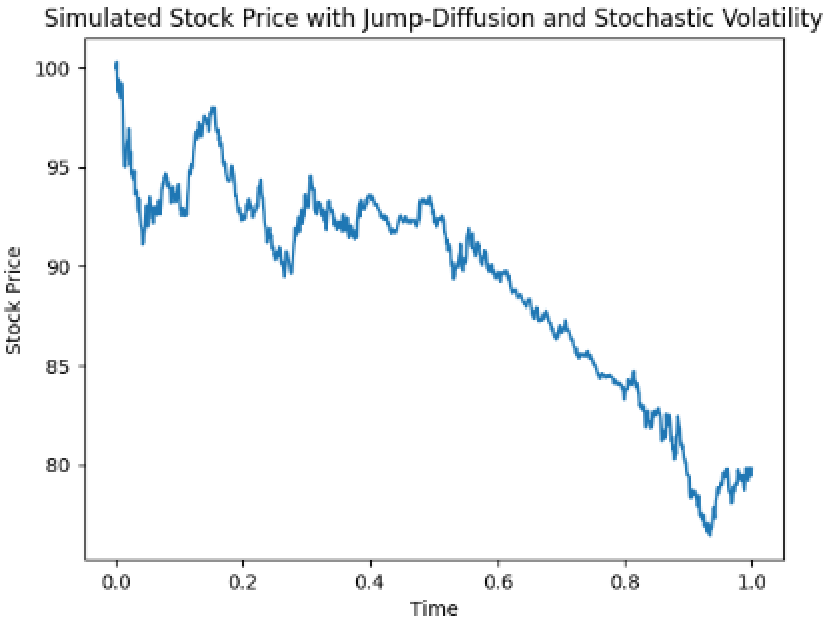
$$dS_t/S_t = (\mu - \beta C_t - \lambda_t k)dt + \sigma_t\, dW_t + J_t dN_t$$

Then we multiply by S_t

Final stock crash model:

$$\boldsymbol{dS_t = S_t\, [(\mu - \beta C_t - \lambda_t k)dt + \sigma_t\, dW_t + J_t dN_t]}$$

We can code this final model into a graph to receive:



The time is in proportion to the number of months in a year, where the stock price on the y axis represents the percent change overall, revealing an accurate graph change of plus or minus 5 percent.

Conclusion

I hope you enjoyed a trip down my mathematical mind, and I presume you can tell its very finance oriented. I know it sounds a bit boring. 'Money this, money that, where's the real math?'

But to me, this is my joy, solving and creating new models to solve potential financial threats that affect everyone worldwide.

As an admirer from afar and up close to calculus, this essay genuinely brought me serenity when deducing variables and equations.

Good thing this essay wasn't volatile, get it? Ok I'll stop.

And finally:

This wouldn't be possible at all without the fathers of calculus, who originated their ideas, Newton and Leibniz. I'm very glad for their existence.

Thank you.