A Collection of Monty Hall Problems

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Introduction

You're on a game show. There are three doors in front of you, and the host announces that there is a car behind only one of those doors and goats behind the others. Your goal is to choose the door with the car, which would seem like a game of pure luck, except you instantly recognise this is the classic Monty Hall problem. This tells you two things. For one, the host is named Monty, and second, after you choose a door, he will open another door that he knows has a goat and offer you a choice to switch. It feels unintuitive to switch as it seems like a 50/50 decision now that one door is eliminated, but you know switching is the best choice and in fact makes you twice as likely to win. You think about why that is.

Suppose you pick Door 1. If the car is behind Door 1, Monty opens either of the remaining doors at random to reveal a goat. If the car is behind Door 2, Monty opens Door 3. If the car is behind Door 3, Monty opens Door 2. Now, suppose Monty opens Door 2. This eliminates the second possibility with the car behind Door 2 as well as a part of the first possibility where the car is behind Door 1 and he opens Door 3 instead. The key insight lies in the forced decision to open Door 2 if the car is behind Door 3, whereas he has a choice to open Door 3 if the car is behind Door 1. You deduce that Monty's knowledge of where the car is favours the possibility of the car being behind Door 3 and switching wins two times out of three.

Despite this informal line of reasoning, you convince yourself to switch, and indeed, the events play out as you imagine. You pick Door 1, Monty opens Door 2, you switch to Door 3, and Monty opens it to reveal a car. You celebrate and think you can leave with the car, but as you later discover, Monty is in fact nowhere near done.

Double or Nothing

Monty offers, or more accurately, forces, for lack of a better word, a second game to test your luck. You pick Door 1 again, but in an interesting turn of events, as he attempts to walk towards and open the door of his choosing, he comically trips on air and falls badly, briefly losing his memory. Instead of waiting to regain his memory, he decides to randomly choose between Door 2 and Door 3, and he ultimately opens Door 2 which happens to contain a goat.

You've also heard of this scenario, coined the Monty Fall problem, and you know in this case switching wins half the time, but why? You revisit the original scenario where Monty is forced to open Door 2 when the car is behind Door 3 but chooses randomly between Door 2 or Door 3 when the car is behind Door 1. However, now that Monty has no knowledge of the car's location, his decision doesn't favour the possibility of the car being behind Door 3 over Door 1. This time, you're not entirely satisfied with your reasoning, so you decide to perform some calculations.

You recall Bayes' theorem, which states that

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

where $P(A|B)$ denotes the probability of event A given B. It might be easier to understand in this form,

$$P(A|B) = \frac{P(B|A)}{P(B)} \cdot P(A)$$

where the fraction acts as a multiplier based on how strongly event B suggests the occurrence of event A. Essentially, it updates the prior probability of event A based on new evidence. In this case, the event is the car being behind Door 3 and the evidence is opening Door… 2 and there being a… goat. Right, you realise your thoughts are getting jumbled up now and it's best to use notation to keep things concise, so you let C_d, G_d, and O_d represent the events of car, goat, and open for each door $d = 1, 2, 3$.

Given that the car is not behind Door 2, the probability of a goat is 1, but if the car is behind Door 2, the probability of a goat is 0. Regardless, the chance of Monty opening Door 2 stays at half, resulting in the following probabilities.

$$P(O_2 \text{ and } G_2 \mid C_1) = \frac{1}{2}$$

$$P(O_2 \text{ and } G_2 \mid C_2) = 0$$

$$P(O_2 \text{ and } G_2 \mid C_3) = \frac{1}{2}$$

You proceed to multiply each probability by one third which is the probability of the car being behind each door, then sum them to get the total probability of opening Door 2 and seeing a goat. Applying Bayes' theorem, you get

$$\begin{aligned} P(C_3 \mid O_2 \text{ and } G_2) &= \frac{P(O_2 \text{ and } G_2 \mid C_3)P(C_3)}{P(O_2 \text{ and } G_2)} \\ &= \frac{P(O_2 \text{ and } G_2 \mid C_3)P(C_3)}{\frac{1}{3}\sum_{d=1}^{3} P(O_2 \text{ and } G_2 \mid C_d)} \\ &= \frac{\frac{1}{2} \cdot \frac{1}{3}}{\frac{1}{3}\left(\frac{1}{2} + 0 + \frac{1}{2}\right)} \\ &= \frac{1}{2} \end{aligned}$$

You also apply this calculation to the first round to ensure you didn't get the equation wrong.

$$\begin{aligned} P(C_3 \mid O_2) &= \frac{P(O_2 \mid C_3)P(C_3)}{P(O_2)} \\ &= \frac{P(O_2 \mid C_3)P(C_3)}{\frac{1}{3}\sum_{d=1}^{3} P(O_2 \mid C_d)} \\ &= \frac{1 \cdot \frac{1}{3}}{\frac{1}{3}\left(\frac{1}{2} + 0 + 1\right)} \\ &= \frac{2}{3} \end{aligned}$$

Comparing the two calculations, you notice that your intuition in pointing out the forced decision was correct since the only term that differs from the additional evidence of goat in Door 2 is $P(O_2 \mid C_3) = 1$ and $P(O_2 \text{ and } G_2 \mid C_3) = \frac{1}{2}$. Monty must open Door 2 when he knows Door 3 contains the car, whereas in the latter case, he could've chosen Door 3, just that it would've revealed a car that he didn't know was there.

Anyway, this round is truly down to luck. You trust your instincts and stick to Door 1, and what do you know, you've now won two cars.

Rigging the Game?

Monty's run out of cars at this point, so instead he proposes a magical crystal orb as your prize for the third round. You have no complaints since you wouldn't know what to do with a third copy of the same car anyway. Monty pulls an unprecedented move and spins a wheel with ratios

4:2:1 to determine which door to put the prize in. He also tells you that the probability distribution is decreasing, so Door 1 is twice as likely as Door 2 which is twice as likely as Door 3 to contain the prize. However, there is a 50% chance that he is lying and might set the probability distribution to be increasing instead. Oh, and he also put you on a time limit. This is getting quite interesting.

You consider choosing Door 2 first since that eliminates the uncertainty of the probability distribution since you know for sure that Door 2 has a $\frac{2}{7}$ chance of hiding the prize. Considering the decreasing geometric sequence, you first calculate the probability of winning by switching if Monty opens Door 3.

$$\begin{aligned}
P(C_3 \mid O_1) &= \frac{P(O_1 \mid C_3)P(C_3)}{P(O_1)} \\
&= \frac{P(O_1 \mid C_3)P(C_3)}{\frac{4}{7}P(O_1 \mid C_1) + \frac{2}{7}P(O_1 \mid C_2) + \frac{1}{7}P(O_1 \mid C_3)} \\
&= \frac{1 \cdot \frac{1}{7}}{\frac{4}{7} \cdot 0 + \frac{2}{7} \cdot \frac{1}{2} + \frac{1}{7} \cdot 1} \\
&= \frac{1}{2}
\end{aligned}$$

You do the same for the case where Monty opens Door 1.

$$\begin{aligned}
P(C_1 \mid O_3) &= \frac{P(O_3 \mid C_1)P(C_1)}{P(O_3)} \\
&= \frac{P(O_3 \mid C_1)P(C_1)}{\frac{4}{7}P(O_3 \mid C_1) + \frac{2}{7}P(O_3 \mid C_2) + \frac{1}{7}P(O_3 \mid C_3)} \\
&= \frac{1 \cdot \frac{4}{7}}{\frac{4}{7} \cdot 1 + \frac{2}{7} \cdot \frac{1}{2} + \frac{4}{7} \cdot 0} \\
&= \frac{4}{5}
\end{aligned}$$

You find that neither case proves staying advantageous, so the optimal strategy in any case is to switch. You also realise there is no need to consider the increasing probability distribution, as the results to the two calculations would simply be swapped.

Out of curiosity, you wonder roughly how often you will win with this strategy across many games, so you calculate the weighted average of the conditional win probabilities given by

$$P(O_1)P(C_3 \mid O_1) + P(O_3)P(C_1 \mid O_3)$$

but wait, from Bayes' theorem, this is equivalent to

$$\begin{aligned}
&P(O_1 \mid C_3)P(C_3) + P(O_3 \mid C_1)P(C_1) \\
&= 1 \cdot \frac{1}{7} + 1 \cdot \frac{4}{7} \\
&= \frac{5}{7}
\end{aligned}$$

That's cool and all, and you're about to perform similar calculations for the remaining two initial choices, but Monty demands an answer as you are nearing the time limit. You have no choice but to choose Door 2, but that's alright since those are pretty good odds. Monty opens Door 1, you switch, and Monty reveals the crystal orb behind Door 3. Unfortunately, the calculations might have to wait another hour, because the game goes on.

Sometimes Psychic

Monty regretfully informs you he lost his other larger crystal orb and instead offers a scientific calculator that looks suspiciously alike to the one you lost days ago. He then hands you the crystal orb from last round and reads the instruction manual, noting that it can predict the unknown with an accuracy of $a = \frac{7}{10}$.

Since he didn't declare if he might be lying, you take it that he's telling the truth, and you follow the orb's prediction and choose Door 1. Monty opens Door 3 to reveal a goat, and you start crunching the numbers. You denote the probability of the prize being at Door 1 given the prediction of Door 1 with $P(C_1 \mid S_1)$ where S stands for psychic.

$$\begin{aligned}
P(C_1 \mid S_1) &= \frac{P(S_1 \mid C_1)P(C_1)}{P(S_1)} \\
&= \frac{P(\mathrm{S}_1 \mid C_1)P(C_1)}{\frac{1}{3}\sum_{d=1}^{3} P(\mathrm{S}_1 \mid C_d)} \\
&= \frac{a \cdot \frac{1}{3}}{\frac{1}{3}\left(a + \frac{1-a}{2} + \frac{1-a}{2}\right)} \\
&= a
\end{aligned}$$

The remaining probability $1 - a$ is evenly distributed across the two unchosen doors since $P(C_2 \mid S_1) = P(C_3 \mid S_1)$. To find the probability of winning by switching, you calculate

$$P(C_2 \mid O_3) = \frac{P(O_3 \mid C_2)P(C_2)}{P(O_3)}$$

$$
\begin{aligned}
&= \frac{P(O_3 \mid C_2)P(C_2)}{aP(O_3 \mid C_1) + \frac{1-a}{2}P(O_3 \mid C_2) + \frac{1-a}{2}P(O_3 \mid C_3)} \\
&= \frac{1 \cdot \frac{1-a}{2}}{a \cdot \frac{1}{2} + \frac{1-a}{2} \cdot 1 + \frac{1-a}{2} \cdot 0} \\
&= 1 - a
\end{aligned}
$$

You discover that for the first time, staying is the best option in this case as accuracy of prediction exceeds 50%, so most likely you've picked the correct option from the start. You lock in your choice, and Monty opens Door 1 to reveal your prize.

Putting it All Together

Seeing how you're an outstanding mathematician with decent luck, Monty presents one final challenge. This time, he offers you a brand-new, huh, what's that? The crew tells Monty there's no more budget for prizes? Well, that's unfortunate, but at least you get a goat with a party hat if you win.

To spice things up, Monty asks an audience member to spin a random wheel to determine the winning door exactly like in the third round, then place the party hat on the goat behind that door. Thankfully, Monty is now in a better mood and decides to tell you that the geometric series of probabilities will be decreasing from Doors 1 to 3. After you choose your door, Monty will open one of the two doors at random, but if he reveals a party hat, you won't even get the chance to decide to stay or switch and you lose everything instantly.

Like last round, you follow the crystal orb and choose Door 1. Monty then randomly chooses to open Door 2, which fortunately does not reveal the party hat, so do you stay or switch? You think for a while, and… wait, what's going on now?

What Now?

You wake up in a hospital bed. You don't quite remember what happened at the end of that… experience. You return home shortly after some checkups and find a car at your driveway, a damaged but otherwise identical car, a crystal orb, a scientific calculator, and a party hat in your room. Was it a dream? Perhaps it is. And perhaps none of what you're seeing right now is real and it's all a hallucination to escape from revising for your upcoming calculus exam. Perhaps you are the goat.

Closing Remarks

Thank you for reading and I hope you found it as fun as I had writing this. The idea comes from my recent discovery of the Monty Fall problem as well as my enjoyment in encountering new trolley problem variants, but the story itself may have been revealed to me in a fever dream… If you enjoyed it, please attempt to solve for some of the results that I couldn't manage to show thoroughly in full detail (mainly the final challenge), and you can refer to the notes and further calculations section below.

References

https://en.wikipedia.org/wiki/Monty_Hall_problem#Other_host_behaviors

https://medium.com/@riat06/intuition-behind-bayes-theorem-ea70bbe863ca

Notes and Further Calculations

The probability of winning by switching for any initial choice can be thought of the sum of the probability of Monty opening one door and the car being behind the other, and vice versa. We can find that $P(A \text{ and } B)$ can be expressed as $P(A|B)P(B)$ or $P(B|A)P(A)$, which is in fact how Bayes' theorem is derived.

If we go back to the third example where the initial location of the prize is determined by a geometric sequence of probabilities, we can find the overall probability of winning by switching without considering the specific condition of which door is opened.

For initial choice of Door 1 and decreasing geometric sequence, the probability of winning is given by

$$P(O_2 \mid C_3)P(C_3) + P(O_3 \mid C_2)P(C_2) = 1 \cdot \frac{1}{7} + 1 \cdot \frac{2}{7} = \frac{3}{7}$$

For increasing geometric sequence, the same calculation is used but yields a value of

$$1 \cdot \frac{4}{7} + 1 \cdot \frac{2}{7} = \frac{6}{7}$$

We then take the average since decreasing and increasing sequences are equally likely to occur, resulting in a final value of $\frac{9}{14}$. Similar calculations prove choosing Door 3 also results in this average win rate, slightly worse than the initial choice of Door 2.

We can also look at the conditional probability of winning given that Monty has opened a door. For initial choice of Door 1 and decreasing geometric sequence, we have

$$
\begin{aligned}
P(C_3 \mid O_2) &= \frac{P(O_2 \mid C_3)P(C_3)}{P(O_2)} \\
&= \frac{P(O_2 \mid C_3)P(C_3)}{\frac{4}{7}P(O_2 \mid C_1) + \frac{2}{7}P(O_2 \mid C_2) + \frac{1}{7}P(O_2 \mid C_3)} \\
&= \frac{1 \cdot \frac{1}{7}}{\frac{4}{7} \cdot \frac{1}{2} + \frac{2}{7} \cdot 0 + \frac{1}{7} \cdot 1} \\
&= \frac{1}{3}
\end{aligned}
$$

$$
\begin{aligned}
P(C_2 \mid O_3) &= \frac{P(O_3 \mid C_2)P(C_2)}{P(O_3)} \\
&= \frac{P(O_3 \mid C_2)P(C_2)}{\frac{4}{7}P(O_3 \mid C_1) + \frac{2}{7}P(O_3 \mid C_2) + \frac{1}{7}P(O_3 \mid C_3)} \\
&= \frac{1 \cdot \frac{2}{7}}{\frac{4}{7} \cdot \frac{1}{2} + \frac{2}{7} \cdot 1 + \frac{1}{7} \cdot 0} \\
&= \frac{1}{2}
\end{aligned}
$$

We can see it is not advantageous to switch in either case of the door opened if the geometric sequence is decreasing. Conversely, if the geometric sequence is increasing, we have

$$
\begin{aligned}
P(C_3 \mid O_2) &= \frac{P(O_2 \mid C_3)P(C_3)}{P(O_2)} \\
&= \frac{P(O_2 \mid C_3)P(C_3)}{\frac{1}{7}P(O_2 \mid C_1) + \frac{2}{7}P(O_2 \mid C_2) + \frac{4}{7}P(O_2 \mid C_3)} \\
&= \frac{1 \cdot \frac{4}{7}}{\frac{1}{7} \cdot \frac{1}{2} + \frac{2}{7} \cdot 0 + \frac{4}{7} \cdot 1} \\
&= \frac{8}{9}
\end{aligned}
$$

$$
\begin{aligned}
P(C_2 \mid O_3) &= \frac{P(O_3 \mid C_2)P(C_2)}{P(O_3)} \\
&= \frac{P(O_3 \mid C_2)P(C_2)}{\frac{1}{7}P(O_3 \mid C_1) + \frac{2}{7}P(O_3 \mid C_2) + \frac{4}{7}P(O_3 \mid C_3)} \\
&= \frac{1 \cdot \frac{2}{7}}{\frac{1}{7} \cdot \frac{1}{2} + \frac{2}{7} \cdot 1 + \frac{4}{7} \cdot 0}
\end{aligned}
$$

$$= \frac{4}{5}$$

In this case, there is an overwhelming advantage to switch, even better than the odds of initial choice of Door 2. However, since we don't know whether the geometric sequence of probabilities is decreasing or increasing, it is a massive gamble if we switch and the expected value of switching is higher for initial choice of Door 2 as proven by the unconditional win rate. Calculations for initial choice of Door 3 will show that the probabilities behave similarly to Door 1 and is ultimately inferior to switching with initial choice Door 2.

Now, it's time to solve the final puzzle.

Starting from the descending geometric sequence case, we have

$$\begin{aligned} P(S_1) &= \sum_{d=1}^{3} P(C_d)P(\mathrm{S}_1 \mid C_d) \\ &= \frac{4}{7} \cdot a + \frac{2}{7} \cdot \frac{1-a}{2} + \frac{1}{7} \cdot \frac{1-a}{2} \\ &= \frac{5a+3}{14} \end{aligned}$$

$$P(C_1 \mid S_1) = \frac{P(S_1 \mid C_1)P(C_1)}{P(S_1)} = \frac{a \cdot \frac{4}{7}}{\frac{5a+3}{14}} = \frac{8a}{5a+3}$$

$$P(C_2 \mid S_1) = \frac{P(S_1 \mid C_2)P(C_2)}{P(S_1)} = \frac{\frac{1-a}{2} \cdot \frac{2}{7}}{\frac{5a+3}{14}} = \frac{2-2a}{5a+3}$$

$$P(C_3 \mid S_1) = \frac{P(S_1 \mid C_3)P(C_3)}{P(S_1)} = \frac{\frac{1-a}{2} \cdot \frac{1}{7}}{\frac{5a+3}{14}} = \frac{1-a}{5a+3}$$

We can then weigh these conditional probabilities based on the new information of Door 2 being opened and no prize being behind it. This is similar to how we have calculated $P(O_2 \text{ and } G_2) = \frac{1}{3}\sum_{d=1}^{3} P(O_2 \text{ and } G_2 \mid C_d)$ in the original Monty Fall problem, except now the probability of each event C_d is different and the conditioning event S_1 is carried through.

$$\begin{aligned} P(O_2 \text{ and } G_2 \mid S_1) &= \sum_{d=1}^{3} P(C_d \mid S_1)P(O_2 \text{ and } G_2 \mid C_d, S_1) \\ &= \frac{8a}{5a+3} \cdot \frac{1}{2} + \frac{2-2a}{5a+3} \cdot 0 + \frac{1-a}{5a+3} \cdot \frac{1}{2} \\ &= \frac{7a+1}{10a+6} \end{aligned}$$

After that, we apply Bayes' theorem, again with the conditioning event carried through, to find the probability of winning by switching.

$$\begin{aligned}
P(C_3 \mid O_2 \text{ and } G_2, S_1) &= \frac{P(O_2 \text{ and } G_2 \mid C_3, S_1)P(C_3 \mid S_1)}{P(O_2 \text{ and } G_2 \mid S_1)} \\
&= \frac{\frac{1}{2} \cdot \frac{1-a}{5a+3}}{\frac{7a+1}{10a+6}} \\
&= \frac{1-a}{7a+1}
\end{aligned}$$

Solving for this probability being bigger than 0.5, we get

$$\begin{aligned}
\frac{1-a}{7a+1} &> 0.5 \\
3.5a + 0.5 &< 1 - a \\
4.5a &< 0.5 \\
a &< \frac{1}{9}
\end{aligned}$$

Therefore, the optimal choice is always to stay as long as the crystal orb's predictions aren't worse than random, which certainly isn't the case in the story, so staying is strongly preferred.

Extra challenge left as an exercise to readers: What happens if like in the third round, there is a 50% chance that the geometric sequence increases instead?