Fourier Transforms: The Pitchlessness of Thunder

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1 Introduction

You're standing on a busy London street on a particularly dreary day, with the sounds of the city swirling around you: the low rumble of cars, the splash of hurried footsteps, the rhythmic drumming of rain, and the chattering of passers-by. To a machine, this would be a meaningless sea of noise. Yet, without a second thought, you perfectly pluck each sound from this frenzied orchestra, as naturally as breathing. The hidden magic behind this remarkable ability is the same reason why thunder has no pitch: the Fourier transform.

2 From Series to Transform

In 1822, the mathematician Joseph Fourier proposed that certain functions, even those which have sharp corners or discontinuities, could be expressed as a series of sine terms. This is known as the *Fourier series*, and it means that complex sound waves can be decomposed into a simple sum of sine functions.

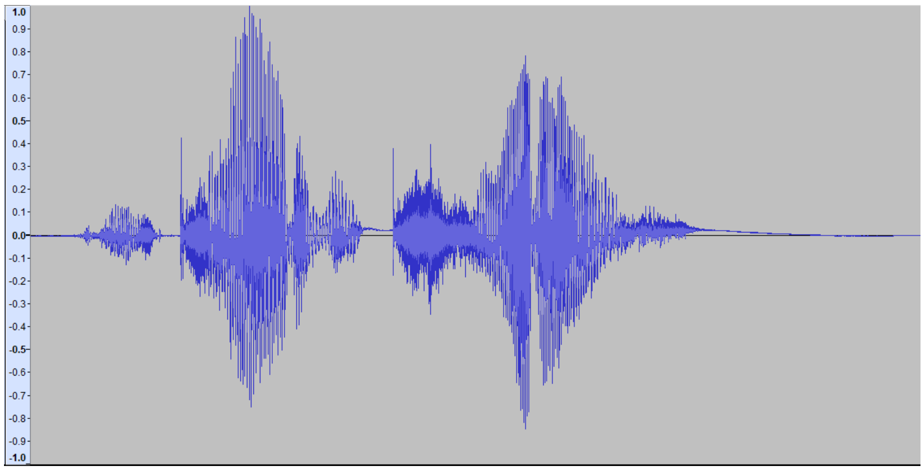


Figure 1: Example of a sharp, discontinuous sound in the time domain which lacks the periodic structure required for the Fourier series.

However, in general, the Fourier series can only be used to express repeating (periodic) functions.

Discrete sounds like a thunderclap or the crack of a whip are clearly not periodic – they don't repeat over time, and so we cannot naturally express them using the Fourier series. This is where the beauty of the *Fourier transform* comes in, which allows us to deal with non-periodic functions. A non-periodic sound signal in the time domain will appear as isolated, irregular spikes, as shown in Figure 1, which is clearly incompatible with the Fourier series. However, we can still analyse the signal by imagining this discontinuous signal to be a single period, T, of a periodic function, and subsequently letting $T \to \infty$. This causes two things to happen: firstly, the discrete frequencies making up the signal get compressed infinitely closely together. Secondly, this therefore causes the sum of these discrete frequency components to naturally transition into a continuous spread of frequencies as they get packed together; what was a countable set of frequencies merges into a continuous spectrum – an integral – *this* is the Fourier transform. In short, instead of picking out individual notes, we now describe the entire range of frequencies present in the sound; rather than asking *when* something happens, the Fourier Transform asks *what* frequencies it contains.

3 The Mathematics of Projection

To understand how this works, we need to ask a deceptively simple question: how do you measure how much two things have in common? Well, one of the first things you learn about in linear algebra is the *dot product* – a simple way of multiplying vectors, mathematical quantities which have both direction and magnitude. If a vector is multiplied by a *unit vector* — a vector with a mag-

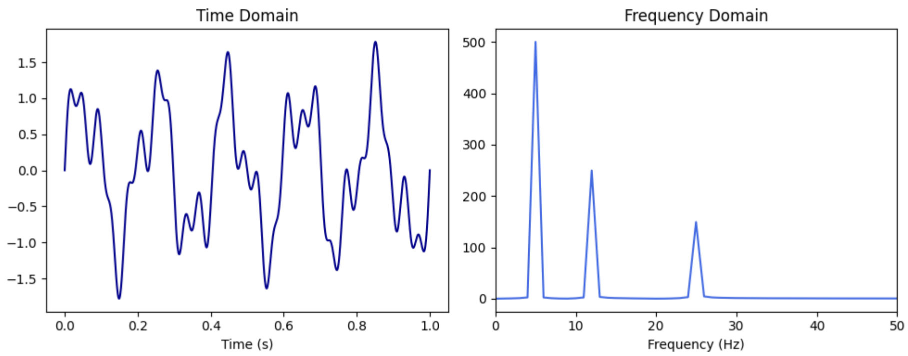


Figure 2: Example of how Fourier Transform decomposes a complex wave into a continuous spectrum of constituent frequencies.

nitude of 1 — using the dot product, the result tells you how much the first vector 'points' in the direction of the unit vector. The *inner product* generalises this idea beyond arrows in space: rather than measuring how much two vectors point in the same direction, it measures what two *functions* have in common — how much they 'overlap'.

But how does this relate to Fourier Transforms? Analogous to the dot product showing how much a vector points in a particular direction, the Fourier Transform, in a special case of the inner product, projects a function, $f(t)$, onto a set of *basis functions* – a set of simple building blocks which can be combined to make any signal. In the context of the Fourier Transform, these basis functions are complex exponentials of the form $e^{i\omega t}$, where i is the imaginary unit, which represent circular motion in the *complex plane* – a 2D coordinate system – at angular frequency ω. These complex exponentials are chosen specifically because they connect rotation to oscillation by Euler's Formula, $e^{i\omega t} = \cos\omega t + i\sin\omega t$. Then, just like the dot product, the Fourier transform measures how much the signal, $f(t)$, and the complex exponential, with frequency ω, 'overlap'.

For a set of basis vectors to be useful, they must be independent (orthogonal) of each other, meaning one frequency won't interfere with another. Remarkably, what we find is that complex exponentials with different values of ω are effectively independent of each other; their inner products (overlap) are zero. This acts as a perfect mathematical filter – when you are testing for part of a signal of a particular frequency, all other components of the signal with different frequencies are ignored. As stated earlier, the transform considers the non-periodic function over all time, from $-\infty$ to ∞, which causes the discrete frequencies to form a continuous spectrum, as shown in Figure 2.

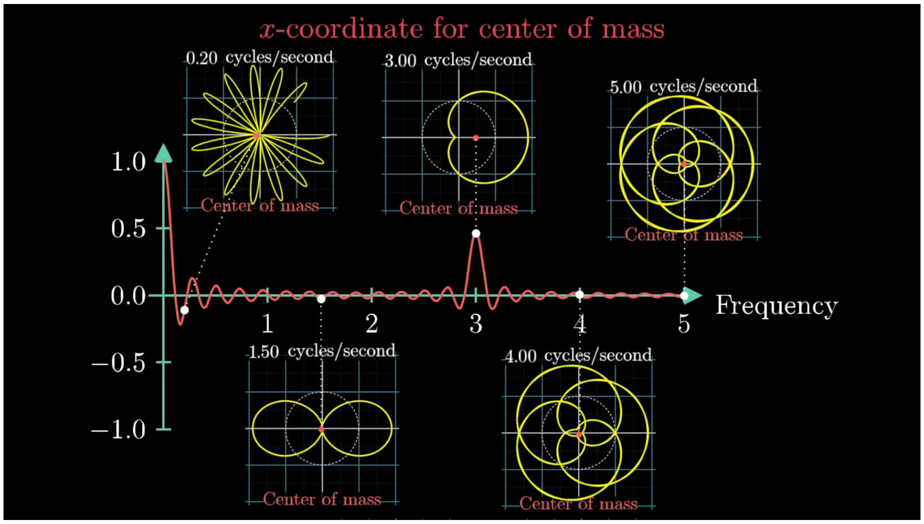


Figure 3: A visual representation of how the wire is 'wound' around the origin at different frequencies. Notice that when the winding frequency matches the signal's frequency of 3 Hz, the centre of mass noticeably shifts away from the origin.

Therefore, the general formula for the Fourier transform is:

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\, dt$$

Let's explain how the Fourier transform actually identifies which frequencies are present in a signal – visually. First, let's express the sharp sound of a thunderclap as a function, $f(t)$. If we consider $f(t)$ to be a long, thin wire with a definite mass, as we compute the Fourier transform, we are effectively 'winding' the function around the origin, since $e^{i\omega t}$ represents rotation at frequency ω. If the frequency of the input signal is different to ω, then the integral averages out to a near-zero value. This is because if the winding frequency is different to the wave's frequency of oscillation, the peaks of the wave end up on *opposite sides* of the circle, 'cancelling' each other out – this is the geometric representation of destructive interference, as shown in Figure 3.

If we calculated the centre of mass of the 'wire' for mismatched frequencies, it would lie perfectly at the origin due to the cancellation of the peaks and troughs. The integral sign in the Fourier Transform is effectively 'summing up' the components of the wave at different points, just as area is summed up in a centre of mass calculation. Conversely, if the winding frequency is *precisely* the same as the signal's frequency something special happens – the peaks become concentrated on exactly the same spot on the circle, since they now occur exactly once every full rotation. This causes the peaks to add together, creating a much larger resultant peak – this is constructive interference – and the exact same thing also happens for the troughs. In this special scenario, the integral outputs a non-zero value as the 'centre of mass' leaps away from the origin. This means that a single, pure tone can be extracted from a chaotic sea of sound.

$\hat{f}(\omega)$ is the output of the integral, representing the signal of $f(t)$ after transformation from the time domain to the frequency domain. $|\hat{f}(\omega)|$ – its magnitude – tells us the intensity of that frequency in the signal, with a larger value corresponding to a greater intensity.

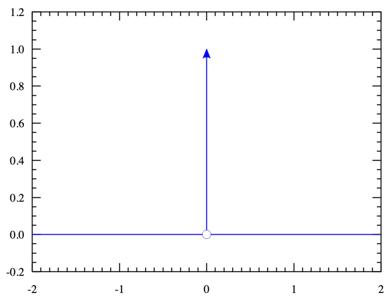
Overall, the Fourier Transform can be thought of as a correlation tool – firstly projecting the signal onto the complex plane, and then identifying which frequencies cause the integral to jump to a non-zero value, revealing the constituent frequencies of the signal.

4 The Duality of Information

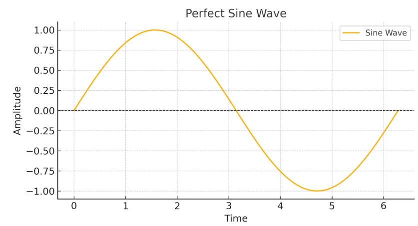
Now that we understand the 'how' of Fourier Transforms, we need to consider why they are possible. If a signal is transformed from the time domain to the frequency domain, no fundamental change of *information* has occurred. The signal itself has no preference for either domain; frequency and time are just two different ways of presenting the same underlying information. This idea is known as the *Duality of Information*.

To illustrate this idea, let's look at the extremes of each case. Imagine a singular sharp spike in the frequency domain stretching upwards to infinity, taking a single defined value. Conversely, if we project this signal onto the time domain, it would appear to be an infinite, perfect sine wave; we have *absolutely no idea* when this sound occurred, since a signal with a pure frequency has an infinite duration.

Now let's consider the other extreme: an infinitesimally sharp sound with near-zero duration, like the near-instantaneous crack of a thunderclap which we mentioned earlier. This sound will have a singular, definitive peak in the time domain – we know almost exactly when the strike happened. However, in the frequency domain, the signal flattens into a horizontal line. This is because the sound was so brief that the concept of pitch itself has completely vanished: the signal's frequency has been forced to spread across the entire spectrum. In other words, the shorter the sound, the less well-defined the pitch becomes. This is the mathematical reason for the pitchlessness of a thunderclap – the sound is so short that it contains every frequency simultaneously; a smear of frequency rather than a distinct note. In gaining an almost perfect 'when' we have utterly sacrificed our 'what'.



(a) Sharp spike in time (Dirac Delta Function) results in a flat frequency spectrum, containing every possible frequency with equal intensity.



(b) Infinite sine wave in the time domain results in a single sharp spike in frequency domain.

Figure 4: The Duality of Information. An instantaneous spike in time (left) requires an infinite sum of all frequencies to exist – the mathematical reason thunder has no pitch. A sound signal over infinite time (right) has a precisely defined frequency.

This may be clearer with a formula which arises from the scaling properties of the Fourier transform: $\Delta f \propto \frac{1}{\Delta t}$, where Δt is the duration of the sound and Δf is the 'bandwidth' of the signal – the difference between its maximum and minimum frequency. As $\Delta t \to 0$ the signal converges to a singular point in time; the bandwidth of the signal must consequently tend towards infinity. By definition the signal now contains *every possible frequency*, since the difference between the maximum and minimum frequencies is now also infinite. The distribution of a signal with an infinitesimally small duration is known as the *Dirac Delta Function*; the Fourier Transform of this signal outputs a constant value for every frequency, as it contains all of them with equal intensity. The idea that Δf grows as Δt shrinks, and vice versa, is a reciprocal relationship colloquially called the *Bandwidth-Duration Theorem*.

5 The Quantum Parallel

In 1924, Louis de Broglie proposed something radical — that all matter behaves like waves. For everyday objects this effect is imperceptibly small, but for minuscule *quantum particles* this phenomenon is unavoidable.

But what does this mean for us? Well, if particles have an intrinsic wave-like property, then they must also be subject to the mathematical constraints of

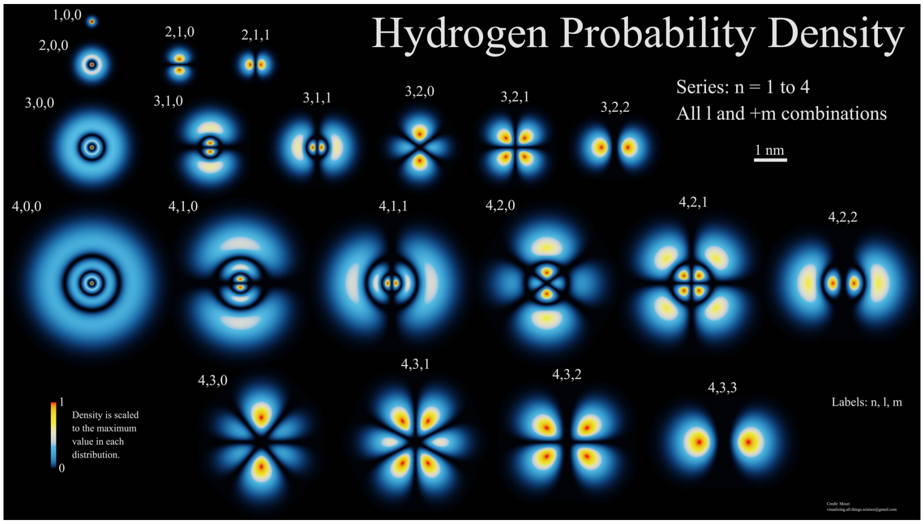


Figure 5: The probability 'cloud' of an electron around a hydrogen atom. Because the electron is a wave, its position and momentum are bound by the same Fourier duality that governs the sound of a London street.

the Fourier Transform. In the quantum landscape, time (t) turns into position (x), and frequency (ω) becomes momentum (p). Just as frequency describes how rapidly a wave oscillates in time, momentum describes how rapidly a quantum wave oscillates in space — this is what physicists mean when they say momentum is 'spatial frequency'. Therefore, the position-space, $\psi(x)$, and the momentum-space, $\phi(p)$, wavefunctions become a *Fourier Transform pair*, just like time and frequency. The duality of information means that these pairs fundamentally behave in mathematically analogous ways. Note that a 'wavefunction' is simply a function which describes the region of space that a quantum particle can occupy.

Considering x and p to be Fourier Transform pairs leads us to an inescapable geometric conflict – if you attempt to confine the position of a particle to a certain point (in a special case of the Dirac delta function), this causes the momentum to 'leak' across the entire spectrum, just as frequency does, taking an infinite number of values – momentum, therefore, is completely uncertain. However, if we attempt to localise the momentum to take a particular value, this causes the position to become a perfect sine wave, just as time does, meaning that we have no idea where the particle is. As strange as it might sound, the particle could be *anywhere* in the entire Universe!

This is the true metaphysical origin of Heisenberg's Uncertainty Principle, $\Delta x \Delta p \geq \hbar/2$, where $\hbar$ is the reduced Planck's constant, which states that there is a maximum level of certainty we can have about a particle's position and momentum. The principle is *not* due to faulty measurement apparatus or a lack of scientific precision – it is a fundamental property of wave-like quantum particles due to the constraints of the Fourier Transform. Thus, we discover that the 'blurriness' of an electron's momentum is due to exactly the same phenomenon as the pitchlessness of a thunderclap – an inescapable trade-off of the Fourier Transform.

6 Conclusion

The Fourier Transform is more than just a clever mathematical trick; it is a lens with which we can discern information from apparent chaos. We live in a world where 'certainty' can never truly be achieved – increased precision in one quantity results in a trade-off with another. To understand the Fourier Transform is to accept that the world around us is not a collection of static objects; it is a dynamic system of constantly interfering and overlapping waves and frequencies. The next time you are sitting on a London bus or standing in the middle of a busy square, know that you are not simply hearing – you are unknowingly performing one of the purest and most fundamental calculations in the Universe.