Let's break Stuff

An essay by Nathan Matlholwa

I have always been fond of breaking stuff. Vases, games, and even my mom's gowns. This love for breaking things has even followed me into mathematics. Unfortunately for me, unlike a vase, math breaks back. One of these unrewarding attempts to break math was based on the idea of divergent geometric sequences, however for you to understand why we have to go through the 3 principles of breaking stuff.

Rule number one of breaking stuff is to understand what you are breaking. Is it made of glass, is it reinforced by something, or does it explode? Luckily for us, it is neither of those. It is a geometric series

Rule n 1: What's this?

What is a Geometric Series

A geometric sequence is all terms of a series added or summed together from the first term of a sequence to some value n, which will serve as the nth term of the sequence, where the sum will stop adding. Better put, we can define a geometric series as such

$$S_n = \sum_{n=1}^{n} ar^{n-1} \quad or \quad S_n = \frac{a(r^n - 1)}{r - 1} \; if \; r \neq 1$$

There are two types of geometric series

- Divergent Geometric Series

- Convergent Geometric Series

A divergent geometric series can be described as a series in which, as we add more and more progressing nth terms of a sequence, the value of the Series never settles at a constant value.

A convergent series, in contrast, is one in which the series converges at a constant value as n approaches infinity. This can be represented as

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(\frac{a(r^n-1)}{r-1}\right) = \frac{a(0-1)}{r-1} = \frac{-a}{r-1} = \frac{a}{1-r}$$

$$\lim_{n\to\infty} S_n = \frac{a}{1-r}$$

Now that we intuitively remember the mechanics of divergent series, let's proceed to Rule number two of breaking stuff: Start breaking!

Let's begin with a simple observation:

Difference of two squares

$(x^2 - a^2) = (x-a)(x+a) \implies (x-a) = (x^{\frac{1}{2}} - a^{\frac{1}{2}})(x^{\frac{1}{2}} + a^{\frac{1}{2}})$

This idea lays the foundation of the mathematical sledgehammer we will use to break the geometric sequence apart. Let's revisit the original geometric sequence

$$S_n = \frac{a(r^n-1)}{r-1}$$

Highlighted in red is the fragment of this series equation we will be interacting with. Established in the previous Difference of two Squares demonstration, we will apply that principle to $(r^n - 1)$ as shown in the series equation

Step 1: Expand $(r^n - 1)$ as follows

$$S_n = \frac{a(r^n - 1)}{r - 1} = \frac{a(r^{\frac{n}{2}} - 1)(r^{\frac{n}{2}} + 1)}{r - 1}$$

This equation is equivalent to the first as established from the previous Difference of two squares axiom we will be using. This is where things get interesting and extremely weird.

Step 2:Expand $(r^n - 1)$ indefinitely i.e to a point $r^{\frac{n}{2^k}}$ where $k \epsilon \mathbb{N}$ for simplicity.

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_n = \frac{a\left(r^{\frac{n}{2^k}} - 1\right)\left(r^{\frac{n}{2^k}} + 1\right)\ldots\left(r^{\frac{n}{2^2}} + 1\right)\left(r^{\frac{n}{2^1}} + 1\right)}{r - 1}$$

In doing this, we are dealing with two infinities, one being 2^k and the other being n.

Dealing with infinities and analyzing how they grow allows us to get an idea of whether a value, namely r, collapses at some point due to this tug of war, so we will evaluate two highlighted brackets of the numerator.

For the duration of this essay we will be treating n and k as elements of natural numbers as well as treating the variable n as a really large number say infinity. This is so that we can apply our findings more generally in the future and have our equations not collapse instantly.

(1) We will first evaluate the first bracket of the expansion, namely $\left(r^{\frac{n}{2^k}} - 1\right)$. This bracket has the following introduced variable 2^k. This variable is meant to represent an expansion of the original bracket (r-1) a k amount of times with k for these purposes being an element of natural numbers.

Let $P_k = \left(r^{\frac{n}{2^k}} - 1\right)\left(r^{\frac{n}{2^k}} + 1\right)\ldots\left(r^{\frac{n}{2^2}} + 1\right)\left(r^{\frac{n}{2^1}} + 1\right)$

In order to evaluate what this limit approaches, we will break the equation down into the following parts

Bracket 1:

We are treating n and k as growing without bound throughout this essay for this reason $\frac{n}{2^k}$ can be represented as $\frac{\infty}{\infty}$. This indeterminate or "unsimplifiable" form that emerges is prompts us to use L'Hôpital's rule in order to assess these two limits simultaneously.

$$\lim_{n\to\infty}\left(\lim_{k\to\infty}\left(\frac{n}{2^k}\right)\right)$$

$$\lim_{k\to\infty}(\frac{n}{2^k}) = \left(\frac{1}{2^k \ln(2)}\right) = \frac{1}{2^k(0.6931)} = \frac{1.4427}{2^k} \approx 0$$

$$\lim_{n\to\infty}(\lim_{k\to\infty}(\frac{n}{2^k})) = \lim_{n\to\infty}(0) = 0$$

It can then be stated that

$$\therefore \lim_{n\to\infty}(\lim_{k\to\infty}(\frac{n}{2^k})) = r^0 - 1 = 1 - 1 = 0$$

Bracket 2:

We will now evaluate the second part of the expression

$$(r^{\frac{n}{2^k}} + 1)(r^{\frac{n}{2^{k-1}}} + 1) \ldots (r^{\frac{n}{2^{3-1}}} + 1)(r^{\frac{n}{2^{2-1}}} + 1)$$

Which simplified can be expresses as follows:

$$\prod_{x=1}^{k}(r^{\frac{n}{2^{k-1}}} + 1)$$

Let us analyse the first bracket

$$\left(r^{\frac{n}{2^k}} + 1\right)$$

The first term $r^{\frac{n}{2^k}}$, as shown in the previous proof, simplifies to r^0. The bracket, hence, can be shown as

$$(r^0 + 1) = 1 + 1 = 2$$

Now, taking the second, third and fourth bracket, etc., will yield us values close to 2, as we will have the following

$$\lim_{k\to\infty}(\frac{n}{2^{(k-1)}}) = \left(\frac{1}{2^{(k-1)} \ln(2)}\right) = \frac{1}{2^{(k-1)}} = \frac{1.4427}{2^{(k-1)}} = 0$$

$$\lim_{k\to\infty}(\frac{n}{2^{(k-2)}}) = \left(\frac{1}{2^{(k-2)} \ln(2)}\right) = \frac{1}{2^{(k-1)}(0.6931)} = \frac{1.4427}{2^{(k-1)}} = 0$$

0 can be seen to be recurring in this limit, and hence we can reasonably deduce that the limit of these equations approaches 0.

$$(2)(2)...(2)(2) = 2^k$$

$$\left(r^{\frac{1000}{2}} + 1\right) = (r^{500} + 1)$$

However, a key contributor to this 2^k value is what we will call the early terms. An example of this early term is the last bracket $\left(r^{\frac{n}{2^1}} + 1\right)$. This first term has a low denominator value of 2, and hence, given a large enough n, such as 1000, could yield a value larger than 1.

$$\left(r^{\frac{1000}{2}} + 1\right) = (r^{500} + 1)$$

This bracket and subsequent following brackets with small exponents for the denominators such a 4,8,16 can hence have form large values which multiply with each other to form a constant value we will term C.

C, is a variable that represents the product of the brackets in the $(r^{\frac{n}{2^k}} + 1)$ ... chain that have an $r^{\frac{n}{2^k}}$ value that is sufficiently larger than 1 hence forming a bracket with adds to a value that is sufficiently larger than 2. These brackets will not be apart of the product 2^k as their values may not be a factor of 2. We term the product of these values C for simplicity.

The early terms which produce C will hence not be a part of the 2^k value we had assumed would form, as they do not form a number close to 2. This would lead us to believe that our late terms, which have values, will only begin at some value at which these early terms begin to shrink to a value close to 2; hence, $2^k$is not the term we should be expressing as the product of all these brackets. The late terms, however, are “infinite” in number, and the finite early terms do not change , 2^k as any finite subtraction of k would simply be absorbed by k as it's an infinitely large value for these purposes.

Taking all this into account, we have the following mathematically

$$\left(r^{\frac{n}{2^k}} + 1\right)\left(r^{\frac{n}{2^{k-1}}} + 1\right) \dots \left(r^{\frac{n}{2^2}} + 1\right)\left(r^{\frac{n}{2^1}} + 1\right) =$$

$$(1 + 1)(1 + 1\ ) \dots \left(r^{\frac{n}{2^2}} + 1\right)\left(r^{\frac{n}{2^1}} + 1\right) =$$

$$(2)(2) \dots (C) =$$

$$(2^k)(C)$$

This simplification allows us to deduce that the "Second" Brackets of P_k produce a product that approaches infinity, indicated by the exponential equation 2^k .

Getting broken

The overall result of our tinkering will produce an indeterminate result of 0*∞.

$$S_n = \frac{a(0)(2^k * C)}{r - 1}$$

This problem arises because we are not literally multiplying by 0 or infinity; we are multiplying (something that is getting really small) by (something that is getting really big). We can alleviate this by applying the technique we used to show bracket 1's limit to evaluate bracket 1 in terms of rate of change i.e. use L'Hôpital's rule.

Breaking Back

Bracket 1:

We use this rule to be able to get our original equation out of an indeterminate form to get an answer we can meaningfully interpret

$$\lim_{k\to\infty}\left(r^{\frac{n}{2^k}}-1\right) = (r^{\frac{n}{2^k}})(\ln(r))(\frac{1}{2^k\ln(2)}) + 0 = \frac{r^{\frac{n}{2^k}}\ln(r)}{2^k\ln(2)}$$

Now that we have a non-zero answer, we can assess the final equation without the problem of having some uninterpretable answer.

$$S_n = \frac{a\left(\frac{(r^{\frac{n}{2^k}})(\ln(r))}{(2^k)((\ln(2))}\right)\color{red}{(2^k)(C)}}{r-1}$$

$$S_n \frac{a\left(\frac{(r^{\frac{n}{2^k}})\,(\ln(r))\color{red}{(2^k)(C)}}{(2^k)(\ln(2))}\right)}{r-1}$$

$$S_n = \frac{\frac{a\left(\left(r^{\frac{n}{2^k}}\right)(\ln r)\color{red}{(C)}\right)}{\ln(2)}}{r-1}$$

This equation, though broken in the sense that it does not produce the standard equation for calculating the sum of a geometric series, is not compelling for me to stop here, so instead, we will use the same methods we applied to get to this point on an neglected part of the equation to get a simple and nonsensical answer to this equation.

$$S_n = \frac{\frac{a\left(\left(r^{\frac{n}{2^k}}\right)(\ln r)(C)\right)}{\ln(2)}}{r-1}$$

The same methods we used to factor the numerator can be applied to the denominator to get an answer that can be interpreted in an easy-to-understand manner. Let's start from the basics, ignoring what we did in the numerator to ensure we do not lose sight of the task at hand.

Hence Let $\frac{a\left(\left(r^{\frac{n}{2^k}}\right)(\ln r)(C)\right)}{\ln(2)} = T_x$

$$S_n = \frac{T_x}{r-1} = \frac{T_x}{(r^{\frac{1}{2}}-1)(r^{\frac{1}{2}}+1)}$$

$$S_n = \frac{T_x}{(r^{\frac{1}{2^k}}-1)(r^{\frac{1}{2^k}}+1)\ldots(r^{\frac{1}{4}}+1)(r^{\frac{1}{2}}+1)}$$

Reusing the same method, we should be able to see a familiar pattern emerging from the denominator. This should hence lead us to a similar answer, hopefully. We will define the two brackets of the equation by the product as follows:

$$A_k = (r^{\frac{1}{2^k}}-1)(r^{\frac{1}{2^k}}+1)\ldots(r^{\frac{1}{4}}+1)(r^{\frac{1}{2}}+1)$$

Bracket 1:

Using L'Hôpital's rule on the first bracket $(r^{\frac{1}{2^k}} - 1)$. We will yield an answer similar to the first bracket of the numerator

$$\lim_{k\to\infty} \left(r^{2^{-k}} - 1\right) = (r^{\frac{1}{2^k}})(\ln (r))((2^{-\mathrm{k}})(\ln (2))(-1))$$

$$= -\frac{\left(r^{\frac{1}{2^k}}\right)(\ln(r))(\ln(2))}{2^k}$$

Bracket 2:

We will reapply the same technique used on Bracket 2 in the numerator to Bracket 2 in the denominator.

$$\left(r^{\frac{1}{2^k}} + 1\right) \ldots \left(r^{\frac{1}{4}} + 1\right)\left(r^{\frac{1}{2}} + 1\right) = 2^k * \mathrm{X}$$

Now you might be wondering, in the numerator we had a value of C instead of X, so why are we are we not using C? The reason we are doing this for now is because the n value in the numerator of $r^{\frac{n}{2}}$ is different from $r^{\frac{1}{2}}$ in the denominator. This means the number of early terms will be reduced if n >1 as the constant 1 approaches 1 at a greater rate.

However, the early terms present in numerator r raised to the n are still present in denominator just without being raised to the n i.e $r^{\frac{n}{2}} = \left(r^{\frac{1}{2}}\right)^n$ This can be used to expression in terms of C in the denominator instead of X. We will call the value of the power of 2 where the early terms end 2^v or completely $r^{\frac{1}{2^v}}$. Where V is the k value where the early terms start or mathematically

$$r^{\frac{n}{2^k}} \approx 1 \approx r^0 \to \frac{n}{2^k} \approx 0$$

Conditions for this to happen with n and k approaching ∞: $2^k \geq n$

$\mathrm{k} \geq \log_2(n)$

Inversely will yield the conditions for early terms hence the conditions for V

$\mathrm{k} < \log_2(n) \rightarrow 1 \leq V < \log_2(n)$

Bracket 2 in the numerator (including only early terms):

$$\left(r^{\frac{n}{2^E}}+1\right)\left(r^{\frac{n}{2^E}}+1\right)\ldots\left(r^{\frac{n}{2^2}}+1\right)\left(r^{\frac{n}{2^1}}+1\right)=C$$

Bracket 2 in the denominator (including only early terms):

$$\left(r^{\frac{1}{2^E}}+1\right)\left(r^{\frac{1}{E}}+1\right)\ldots\left(r^{\frac{1}{2^2}}+1\right)\left(r^{\frac{1}{2^1}}+1\right)=$$

$$\left(\left(r^{\frac{n}{2^E}}+1\right)\left(r^{\frac{n}{2^E}}+1\right)\ldots\left(r^{\frac{n}{2^2}}+1\right)\left(r^{\frac{n}{2^1}}+1\right)\right)^{n^{-1}}=$$

$$\left(\left(r^{\frac{n}{2^E}}+1\right)\left(r^{\frac{n}{2^E}}+1\right)\ldots\left(r^{\frac{n}{2^2}}+1\right)\left(r^{\frac{n}{2^1}}+1\right)\right)^{n^{-1}}=C^{n^{-1}}$$

This expression of C in the denominator disregards X or when the early terms of numerator end as they amount of early terms differ hence rather we will not account for the early terms of the denominator and just assess the equation as in terms of the early terms of the numerator. This is done to help simplify the equation.

Bracket 2 as

$$2^k * C^{n^{-1}}$$

Combining brackets 1 and 2 of the denominator, we get

$$A_k = -\frac{\left(r^{\frac{1}{2^k}}\right)(\ln(r))(\ln(2))}{2^k}(2^k)(C^{n^{-1}})$$

$$A_k = -(\left(r^{\frac{1}{2^k}}\right)(\ln(r))(\ln(2)))(C^{n^{-1}})$$

Bringing it together

$$S_n = \frac{\dfrac{a\left(\left(r^{\frac{n}{2^k}}\right)(\ln r){\color{red}(C)}\right)}{\ln(2)}}{-(\left(r^{\frac{1}{2^k}}\right)(\ln(r))(\ln(2)))(C^{n^{-1}})}$$

$$=a\left(\left(r^{\frac{n}{2^k}}\right)(\ln r){\color{red}(C)}\right)\left(-\left(\left(r^{\frac{1}{2^k}}\right)(\ln(r))\right)(C^{n^{-1}})\right)$$

$$=-a\left(r^{\frac{1}{2^k}}\right)\left(r^{\frac{n}{2^k}}\right)(\ln(r))^2\left(C^{1+\frac{1}{n}}\right)$$

$$S_n = -a\left(r^{\frac{n+1}{2^k}}\right)(\ln(r))^2(C^{\frac{n-1}{n}})$$

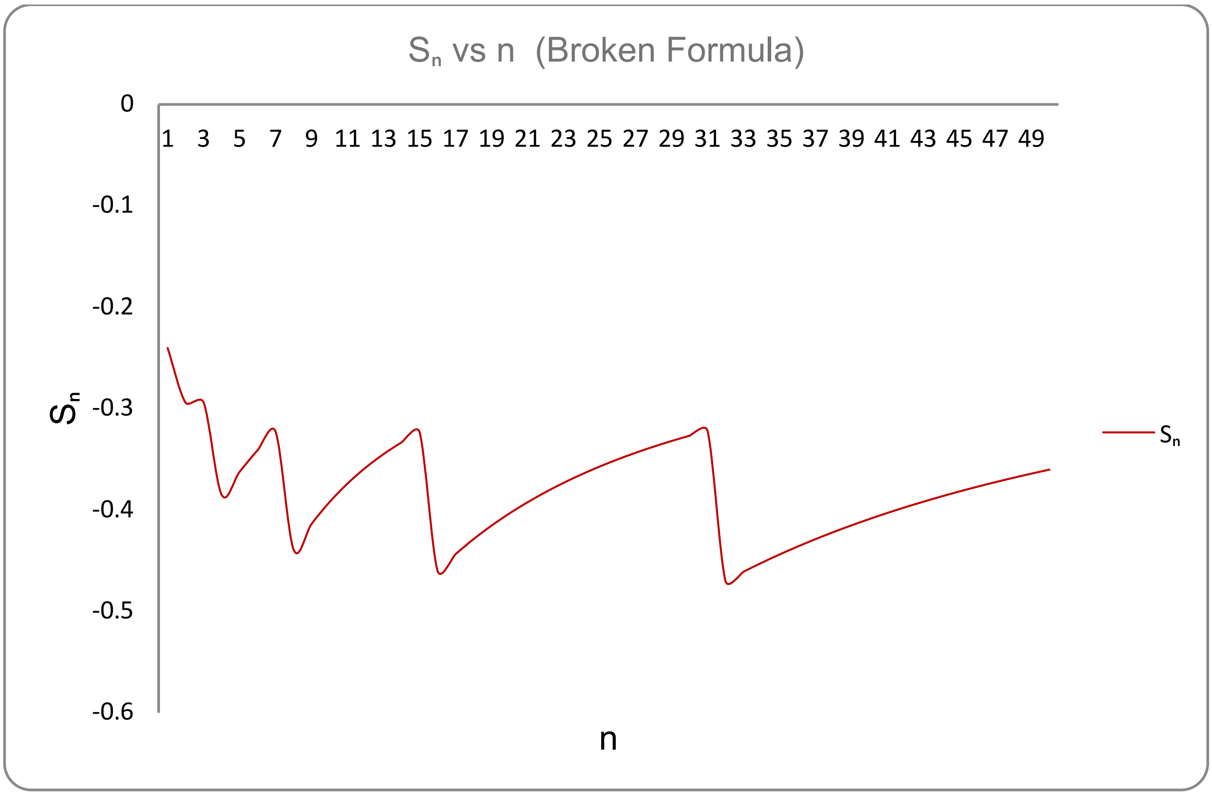
This equation is close to the final answer however there still exists $\frac{n+1}{2^k}$ as we established that would approach 0 hence we can simplify that part of the expression to 1

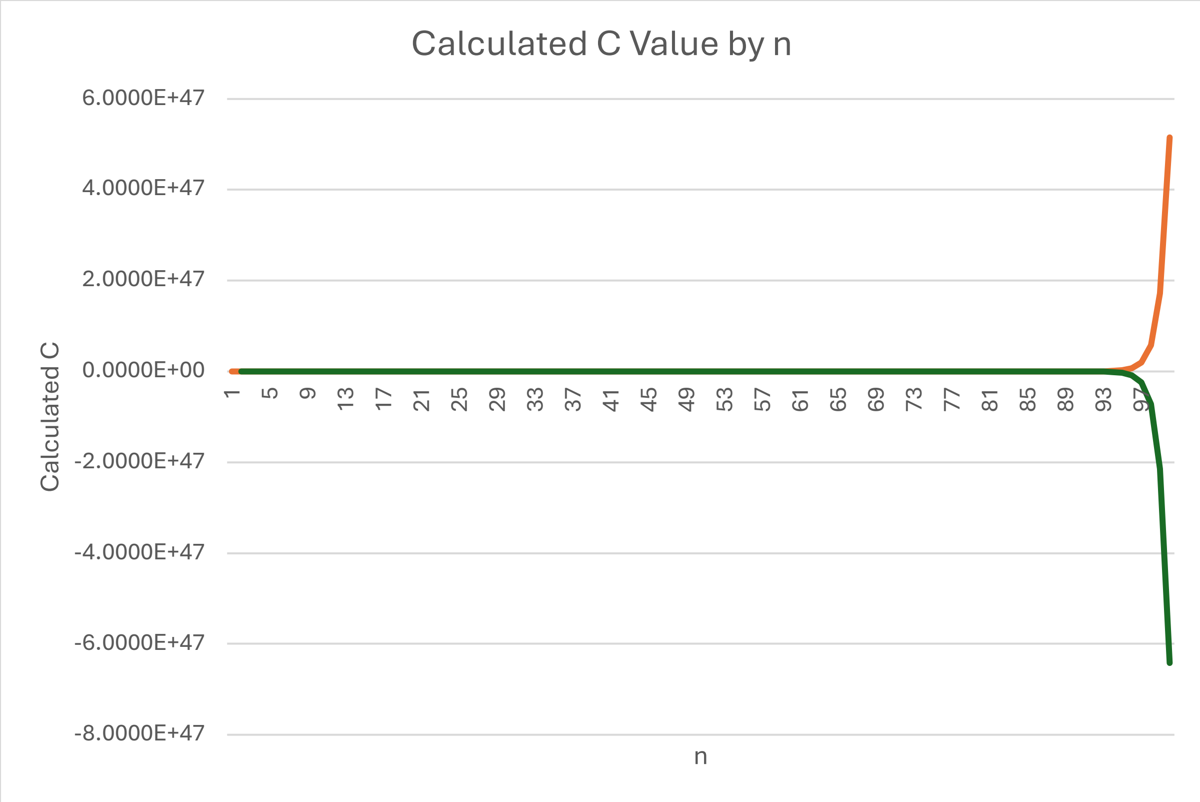
$$S_n = -a(r^0)(\ln(r))^2(C^{\frac{n-1}{n}}) = -a(\ln(r))^2(C^{\frac{n-1}{n}})$$

We have now successfully broken down the equation into an almost unrecognizable version of itself. This serves as our queue to proceed the final stage of breaking stuff.

Rule n 3: Pick up the pieces

This is the saddest part of this journey. Admitting you did indeed break it. This is not meant to be a discouraging stage but rather it is the most important of them all viewing the flaws in the math to be able to reconvene later. We will compare the actual S^n value with our broken equation by graphing the two and trying to notice any relationships. Specifically the constant C appears to depend on the itself, this means our broken formula may not be a true alternative representation





These relationships though misty are hard to spot and would require an essay of their own to solidify mathematically. This aspect of breaking stuff is the least rewarding short term as it may not solve the problem right when you start it, but like all things math that's what makes it fun. I hope you can spot any problems I would have missed with my reasoning and apply them to your own ill mannered equations.