The Function That Broke Every Calculator I Own

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1 The Race

Which is bigger: 2^3 or 3^2? Easy enough. $2^3 = 8$ and $3^2 = 9$. The bigger number wins.

Now try 3^4 versus 4^3. That is 81 against 64. This time the bigger *exponent* wins instead. What about 4^5 versus 5^4? We get 1024 versus 625, and the gap is widening. By the time you reach 10^{11} versus 11^{10}, the score is roughly 100 billion to 26 billion.

A pattern is forming. When x gets larger, x^{x+1} seems to be larger than $(x+1)^x$. But I wanted to understand this pattern more into details, specifically by *how much* they differ. So I wrote down the function

$$f(x) = \frac{x^{x+1}}{(x+1)^x} \tag{1}$$

Think of it as a scoreboard. When $f(x) = 1$, the two sides are tied. When $f(x)$ is large, x^{x+1} is winning, and vice versa. Here are the first few integer values:

x	1	2	3	5	10	50	100	140
$f(x)$	0.50	1.19	1.90	3.29	7.27	38.34	77.42	108.8

The values grow steadily, and seem to have a relation. You don't have to pull out your devices and start plotting. I've done it for you.

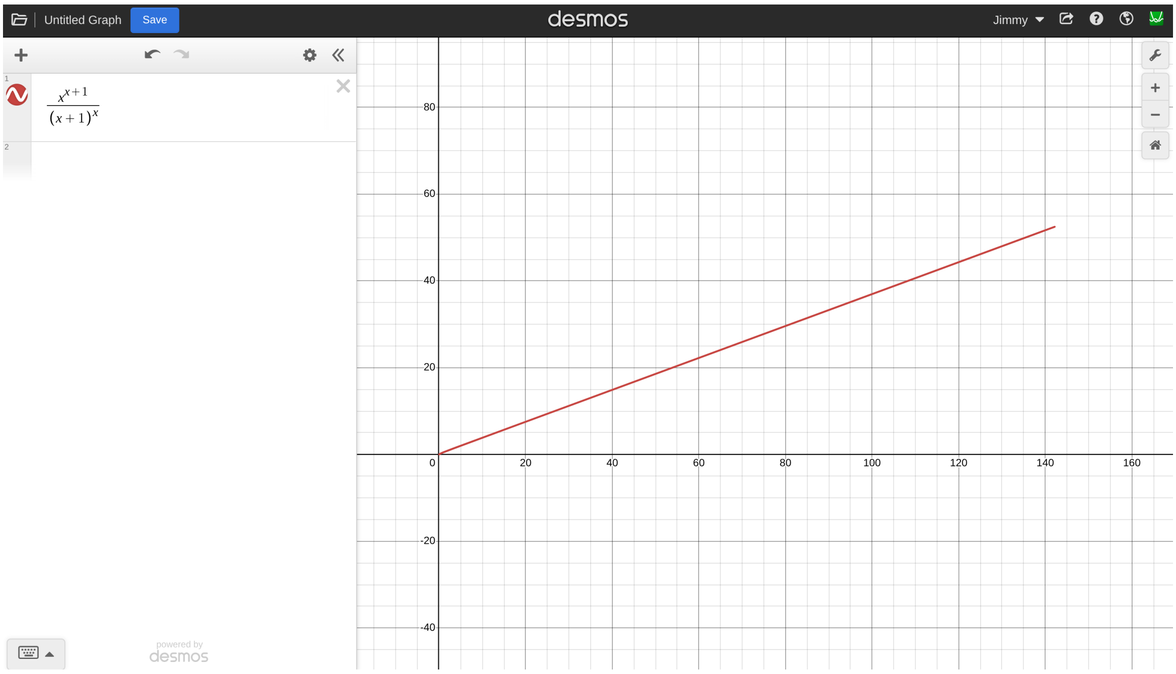

Figure 1: Desmos

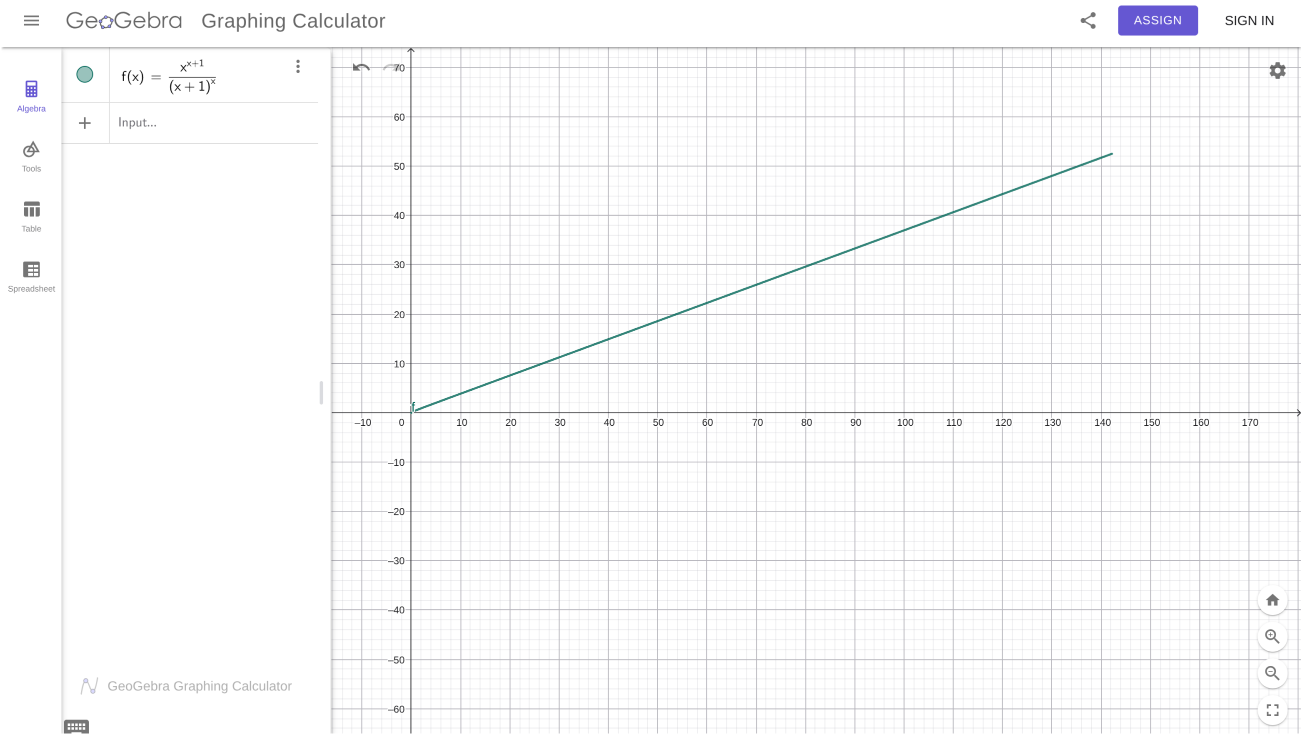

Figure 2: Geogebra

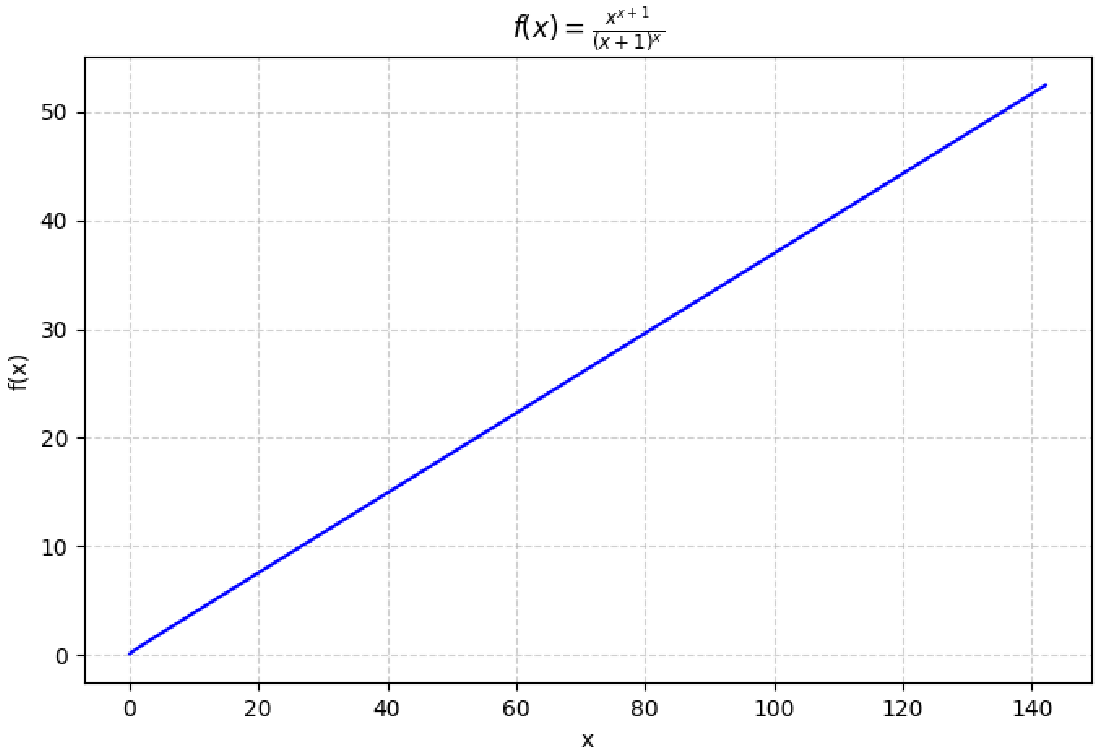

Figure 3: Matplotlib

Desmos, GeoGebra, and Python's Matplotlib all drew the curve beautifully from $x = 0$ to around $x = 140$. And then, without warning, the graph simply *vanished*. There's not a single error message. It just stopped.

Had I discovered a function that *ceases to exist* past a certain point? Was this some exotic discontinuity? Or, less glamorously, was every calculator I own simply giving up?

2 Two Ends

The function has two remarkably different personalities: one near the origin, and one far from it. Studying both will be the key to the mystery.

Near zero

Though might not be obvious in the figures, the function appears to curve near the origin, and starts to merge to a straight line starting from some small x.

To understand the behaviour near zero, I want to find the slope of $f(x)$ at the origin. The key idea is to take the natural logarithm of $f(x)$. This turns products into sums and powers into multiplications, making the expression far easier to handle.

We can split off one factor of x:

$$f(x) = \frac{x \cdot x^x}{(x+1)^x} = x \cdot \left(\frac{x}{x+1}\right)^x$$

Taking the natural logarithm:

$$\ln f(x) = \ln x + x[\ln x - \ln(x+1)] \tag{2}$$

Once we have $\ln f(x)$, we can apply a Taylor series. It is a way of approximating a function near a point using the sum of many polynomial of increasing degree. Like replacing a curved road with a straight one over a short stretch, the more polynomials I use, the more accurate it is.

For small x, the Taylor series gives

$$\ln(x+1) = 0 + x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

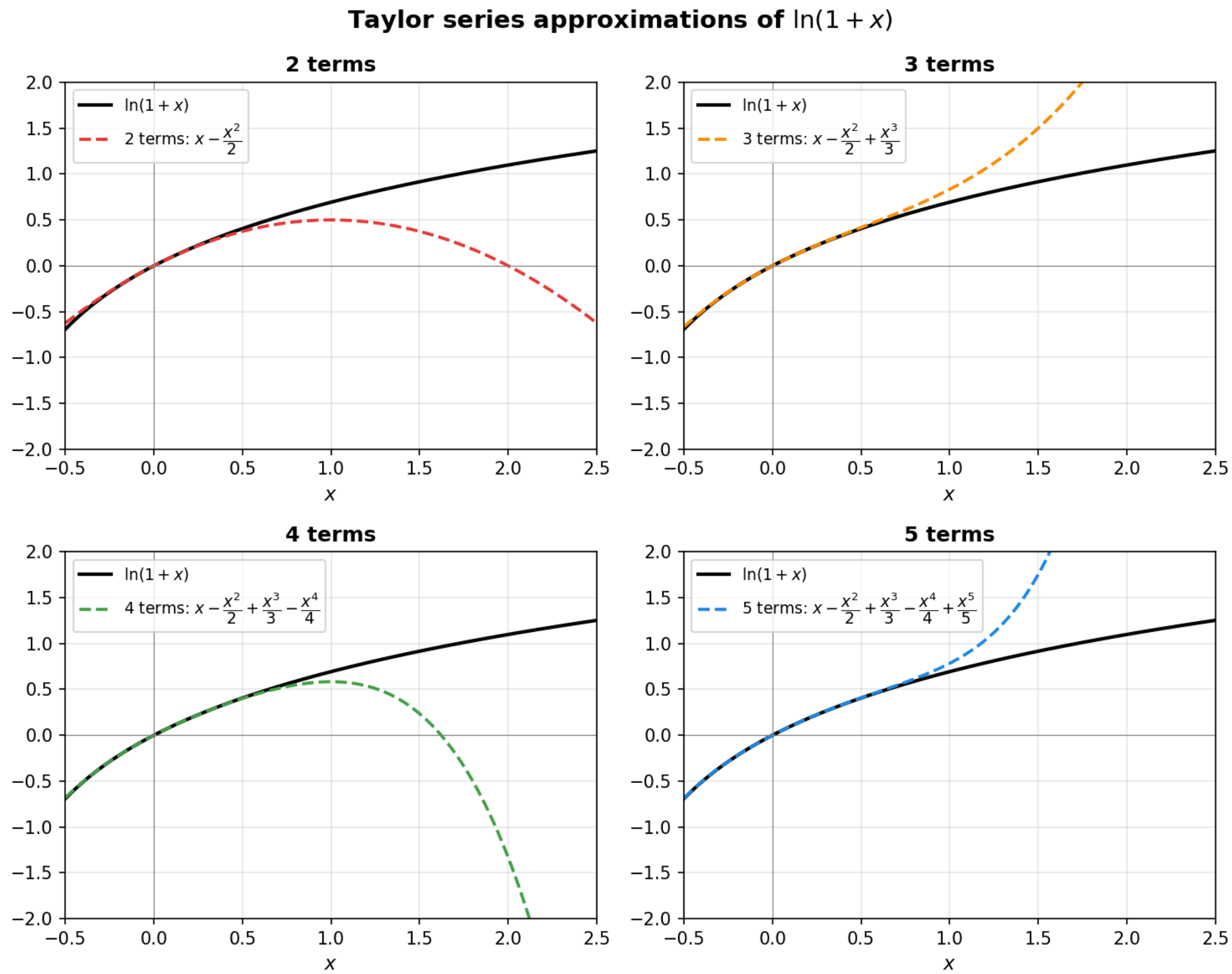

Figure 4: Demonstration of more accurate result as no. of terms increase

This turns (2) into:

$$\begin{aligned}\ln f(x) &= \ln x + x\left[\ln x - x + \frac{x^2}{2} - \cdots\right] \\ &= \ln x + x\ln x - x^2 + \frac{x^3}{2} - \cdots\end{aligned}$$

Exponentiating:

$$f(x) = e^{\ln x + x\ln x\, -\, x^2\, +\, \frac{x^3}{2} - \cdots} = x \cdot e^{x\ln x\, -\, x^2\, +\, \frac{x^3}{2} - \cdots}$$

Now I can find the tangent line at $x = 0$ with first principle:

$$f'(0) = \lim_{x\to 0^+} \frac{f(x) - 0}{x - 0} = \lim_{x\to 0^+} e^{x\ln x - x^2 + \frac{x^3}{2} - \cdots}$$

The notation $x \to 0^+$ simply means that x is approaching zero from the positive side. Imagine x having an initial value of, for example, 100. That x value is decreasing and decreasing, aiming for zero. As x approaches zero, the value of each single term at the exponent will also approach zero, since they're all multiplication with x. Thus the limit becomes:

$$f'(0) = \lim_{x\to 0^+} e^{x\ln x - x^2 + \frac{x^3}{2} - \cdots} = e^0 = 1$$

This tells us that the gradient of the tangent line at $x = 0$ is 1.

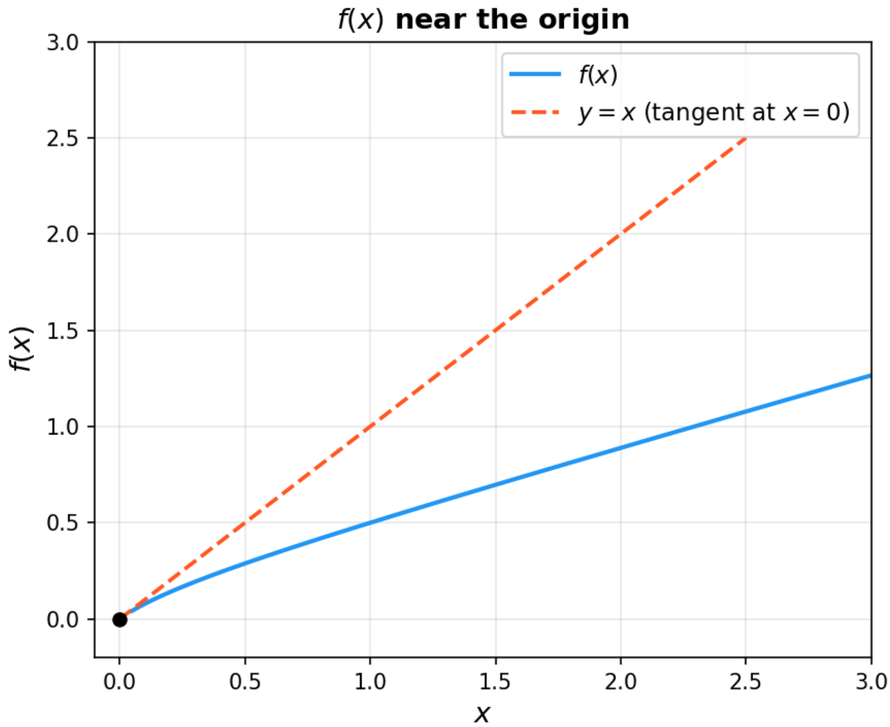

Figure 5: Plot of the tangent at $x = 0$

Therefore, at $x = 0$, $f(x)$ has a tangent line of $y = x$. That is its first "personality". Everything seems normal at this point.

Far from zero: The second personality

For large x, the same logarithm trick applies, but now we expand around a different point. We cannot expand $\ln(x+1)$ for large x because Taylor series will fail. Instead, we rewrite the expression to expand $\ln(1 + 1/x)$, which makes $1/x$ small when x is large. But the idea is the same.

From (2), we have:

$$\begin{aligned}
\ln f(x) &= \ln x + x\ln x - x\ln(x+1) \\
&= \ln x + x\ln x - x\ln\left[x\left(1+\frac{1}{x}\right)\right] \\
&= \ln x + x\ln x - x\left[\ln x + \ln\left(1+\frac{1}{x}\right)\right] \\
&= \ln x - x\left[\ln\left(1+\frac{1}{x}\right)\right]
\end{aligned}$$

We apply the similar method, using Taylor series:

$$\ln\left(1+\frac{1}{x}\right) = 0 + \frac{1}{x} - \frac{1}{2x^2} + \frac{1}{3x^3} - \cdots$$

Substituting into $\ln f(x)$:

$$\begin{aligned}
\ln f(x) &= \ln x - x\left(\frac{1}{x} - \frac{1}{2x^2} + \frac{1}{3x^3} - \cdots\right) \\
&= \ln x - 1 + \frac{1}{2x} - \frac{1}{3x^2} + \cdots
\end{aligned}$$

Exponentiating both sides:

$$f(x) = \frac{x}{e}\cdot\exp\left(\frac{1}{2x} - \frac{1}{3x^2} + \cdots\right). \tag{3}$$

Let $u = \frac{1}{2x} - \frac{1}{3x^2} + \cdots$. As $x \to \infty$, each term in u approaches zero. Keeping the numerator still, as we increase the denominator, the value of the fraction decreases. As the denominator gets infinitely large, the value of the fraction decreases to infinitely small, though not negative, which is zero. This makes $u \to 0$, so we may use the Taylor series:

Since

$$e^x = 1 + x + \frac{x^2}{2} + \cdots$$

We substitute u. To make our lives easier, we may write all terms that has x^{-2} and above together. Since x is a large value, we believe that these terms get very very small. Thus we have:

$$\begin{aligned}
e^u &= 1 + \left(\frac{1}{2x} - \frac{1}{3x^2} + \cdots\right) + \frac{1}{2}\left(\frac{1}{2x} - \cdots\right)^2 + \cdots \\
&= 1 + \frac{1}{2x} - \frac{1}{3x^2} + \frac{1}{8x^2} + \cdots \\
&= 1 + \frac{1}{2x} + \cdots
\end{aligned}$$

Substituting back into (3) and distributing the product:

$$\begin{aligned}
f(x) &= \frac{x}{e}\left(1 + \frac{1}{2x} + \cdots\right) \\
&= \frac{x}{e} + \frac{1}{2e} + \cdots
\end{aligned}$$

Therefore, as $x \to \infty$, the terms in $\cdots$ approach zero. Thus:

$$\boxed{f(x) \sim \frac{x}{e} + \frac{1}{2e}}$$

The asymptote is a straight line with gradient $1/e$ and y-intercept $1/(2e)$.

But pause here. The number e appears *nowhere* in $x^{x+1}/(x+1)^x$. Yet as x grows, the ratio settles into a straight line governed entirely by Euler's number.

But *why*? Where does e actually come from?

3 The ghost of e

The appearance of e is not a coincidence. It is the direct consequence of a limit that many readers will recognise from school:

$$\lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = e \tag{4}$$

Watch what happens when I rewrite $f(x)$:

$$f(x) = x \cdot \left(\frac{x}{x+1}\right)^x = x \cdot \frac{1}{\left(\frac{x+1}{x}\right)^x} = \frac{x}{\left(1 + \frac{1}{x}\right)^x}$$

That denominator is exactly the expression $(1 + 1/x)^x$ that defines e in the limit. So for large x:

$$f(x) \approx \frac{x}{e}$$

This is why e appears. The function $f(x)$ is secretly a dressed-up version of the limit form (4). Every time you compute $f(x)$ for some large value, you are measuring how close $(1 + 1/x)^x$ is to e. The asymptote $y = x/e$ is not some algebraic accident.

The $1/(2e)$ correction in the intercept comes from the *next* term in the asymptotic expansion of $(1 + 1/x)^x$. Using our earlier series more carefully:

$$\left(1 + \frac{1}{x}\right)^x = e^{x\ln(1+\frac{1}{x})} = e^{1-\frac{1}{2x}+\cdots} \approx e\left(1 - \frac{1}{2x}\right)$$

So the denominator is not quite e, but $e(1 - \frac{1}{2x})$. Thus:

$$\frac{x}{e\left(1 - \frac{1}{2x}\right)} = \frac{x}{e} \cdot \frac{1}{1 - \frac{1}{2x}}$$

Let $u = 1/(2x)$, the geometric series tells us:

$$\frac{1}{1-u} = 1 + u + u^2 + \cdots$$

Note that the formula for geometric series only applies when $|u| < 1$. We have established that x is a big number approaching infinity. This makes u a small number approaching zero. Substituting back gives:

$$\frac{x}{e} \cdot \left(1 + \frac{1}{2x} + \cdots\right)$$

Multiply x/e with every term in the brackets, we have:

$$\frac{x}{e} \cdot \left(1 + \frac{1}{2x} + \cdots\right) \approx \frac{x}{e} + \frac{1}{2e}$$

which is exactly the asymptote.

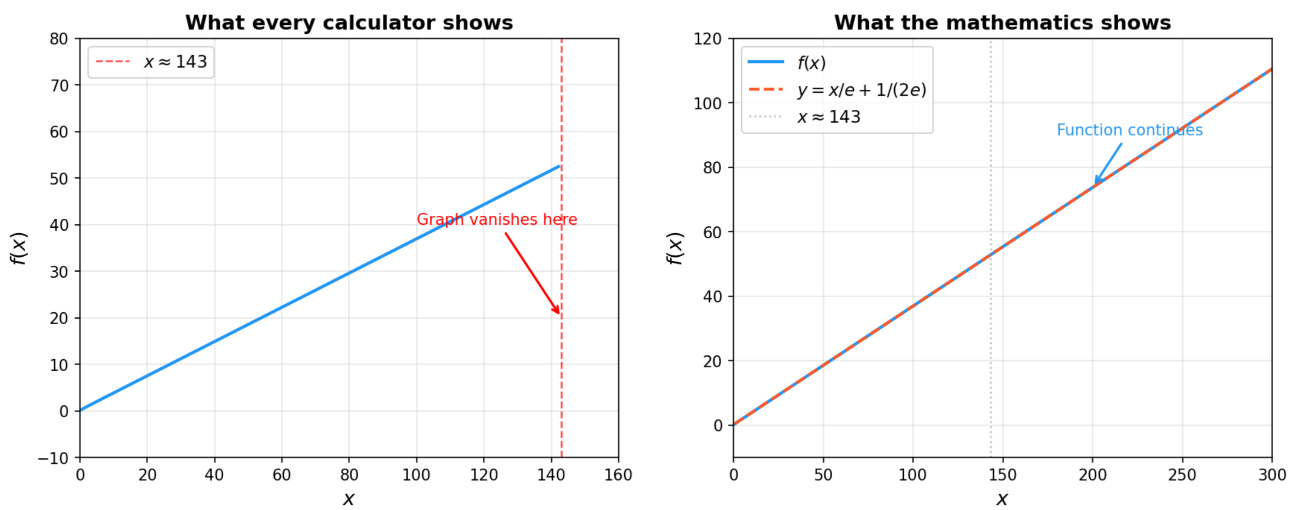

Figure 6: Comparison with & without asymptote

4 A hidden connection to factorials

The link goes deeper still. Consider **Stirling's approximation**, one of the great formulas in mathematics:

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n \quad \text{as } n \to \infty$$

This says that for large n, factorials are governed by the expression $(n/e)^n$. Now look at the ratio of consecutive Stirling approximations:

$$\frac{(n+1)!}{n!} = n+1 \quad \Longrightarrow \quad \frac{\sqrt{2\pi(n+1)}\left(\frac{n+1}{e}\right)^{n+1}}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n} \sim n+1$$

Simplify the left side. The $\sqrt{2\pi}$ gets cancelled, the square roots contribute a factor tending to 1, and we are left with:

$$\begin{aligned}
\frac{\frac{(n+1)^{n+1}}{e^{n+1}}}{\frac{n^n}{e^n}} &\sim n+1 \\
\frac{\frac{(n+1)^{n+1}}{e^n \cdot e}}{\frac{n^n}{e^n}} &\sim n+1 \\
\frac{\frac{(n+1)^{n+1}}{e}}{n^n} &\sim n+1 \\
\frac{(n+1)^{n+1}}{e \cdot n^n} &\sim n+1 \\
\frac{1}{e} \cdot \frac{(n+1)^{n+1}}{n^n} &\sim n+1 \\
\frac{1}{e} \cdot \frac{(n+1)^n}{n^n} &\sim 1 \\
\frac{e \cdot n^{n+1}}{(n+1)^n} &\sim n
\end{aligned}$$

Rearranging:

$$\boxed{\frac{n^{n+1}}{(n+1)^n} \sim \frac{n}{e}}$$

That left-hand side is precisely $f(n)$! (exclamation mark, not factorial) So the asymptotic behaviour $f(x) \sim x/e$ is not merely a curious property of one function. The fact that the graph looks like a straight line with slope $1/e$ is the same fact that makes $n!$ grow like $(n/e)^n$.

5 Proof that the graph never stops

With the asymptote established, I could test whether $f(x)$ truly "stops." If $f(x)/g(x) \to 1$ where $g(x) = \frac{x}{e} + \frac{1}{2e}$, the function lives forever. Substituting $n = x + 1$, $g(x) = \frac{n-1}{e} + \frac{1}{2e} = \frac{n-\frac{1}{2}}{e}$.

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{n\to\infty} \frac{\frac{(n-1)^n}{n^{n-1}}}{\frac{n-\frac{1}{2}}{e}} = \lim_{n\to\infty} e\,\frac{(n-1)^n}{n^{n-1}(n-\frac{1}{2})} = \lim_{n\to\infty} e \cdot \frac{n}{n-\frac{1}{2}} \cdot \left(1 - \frac{1}{n}\right)^n = e \cdot 1 \cdot \frac{1}{e} = 1$$

The function never stops, never breaks, never diverges. Those three graphs were all **wrong** past $x \approx 140$.

6 Why computers choke

Most software uses **double-precision floating-point arithmetic**, which can represent numbers up to roughly 10^{308}, i.e. $\ln(\max) \approx 709$. The numerator x^{x+1} overflows when $(x+1)\ln x > 709$, which happens at $x \approx 143$. The denominator overflows at nearly the same moment. The computer attempts ∞/∞, returns `NaN`, and the graph disappears.

At $x = 143$, the true value of $f(x)$ is approximately $143/e \approx 52.6$. But to compute it, you must first construct two numbers each exceeding 10^{308}, only to divide them and get 52.6. It is like weighing an elephant by first lifting it onto the scale. The measurement itself is simple, but the lifting defeats you.

7 The fix

Work in logarithmic space:

$$f(x) = e^{\,(x+1)\ln x \;-\; x\ln(x+1)}$$

Both terms in the exponent grow like $x \ln x$, which stays within double precision until x is astronomically large. The graph now extends effortlessly past $x = 1{,}000{,}000$.

8 What about other races?

A natural question: what if the exponent gap is not 1 but some other integer k? Define:

$$f_k(x) = \frac{x^{x+k}}{(x+k)^x}$$

Our original function is f_1. Repeating the same asymptotic analysis:

$$f_k(x) = \frac{x^k}{(1+k/x)^x} \;\longrightarrow\; \frac{x^k}{e^k} \quad \text{as } x \to \infty$$

So $f_2(x) \sim x^2/e^2$, $f_3(x) \sim x^3/e^3$, and in general the asymptote is a *polynomial* of degree k with leading coefficient $1/e^k$. The linear case $k = 1$ was special only because it gave a straight line. For larger k, the scoreboard becomes a parabola, a cubic, and so on. Every member of this family breaks every graphing tool at the same point ($x \approx 143$), and every one is rescued by the same logarithmic trick.

9 What I learned

I started this essay because I believed I had found a bug in three separate pieces of software. In a way, I had, but not the kind I expected. The bug was not in the software's logic.

Mathematics told me the function was fine. If it's analytic, it is and should be continuous and smooth, and it should look that way as well. Computation told me the function was broken. What resolved the conflict was in fact *analysis* itself, including Taylor series, limits, and asymptotes. A graph is a picture of an answer. The mathematics is the answer itself.

Along the way I stumbled into something I did not expect. A function I invented to settle a random idea turned out to connect the definition of e, Stirling's approximation, and a family of overflowing functions. None of this was visible on any screen. All of it was on paper.

That, I think, is quite a good reason to keep doing mathematics by hand, even in 2026.

References

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