Not all infinities are equal: A journey through Cantor's theory of infinite sets

In everyday language, we often see infinity as a single, endless idea, the unreachable end of the number line, the limitless universe. It seems like infinity is the same everywhere, uniform and absolute. However, this intuition, while it feels natural, is actually incorrect.

In the late nineteenth century, the German mathematician Georg Cantor (1845-1918) created a solid mathematical framework for comparing the sizes of infinite sets. His discovery was groundbreaking. Not all infinite collections are the same size. Infinities can be larger than one another, and there are countless distinct sizes of infinity, forming a complex hierarchy. This essay will illustrate this claim, starting with the traditional view of infinity that Cantor challenged. It will discuss the key concepts of bijection and cardinality, present the famous diagonal argument as its centre, and wrap up with exceptional open questions that Cantor's framework could not answer.

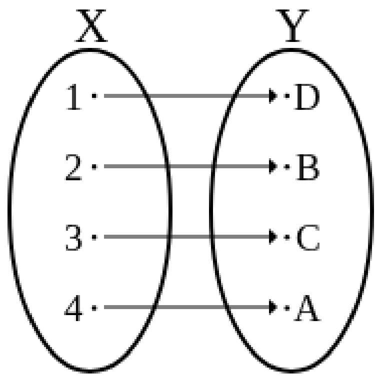
The historic view of infinity

Before Cantor, Mathematicians had no sufficient way of differentiating between infinite sets. The ancient Greek philosopher Aristotle drew a distinction between and potential infinity (a process that proceeds without end such as counting) and an actual infinity (a complete and endless totality). Aristotle was largely suspicious of the latter, and for centuries to come, his caution would ripple through mathematics. Infinity was something to be approached, but never fully grasped as a completed object.

The problem with treating infinity as an object become increasingly apparent in the work of Galileo Galilei. In his 1638 *Discourses,* Galileo observed that every natural number can be squared with its pair, 1 with 1, 2 with 4, 3 with 9, and so on. The pairing never faulters, yet the perfect squares- 1, 4, 9, 16, 25 and so on, see far less common than the natural numbers from which they are extracted. The squares are a proper subset of the naturals, yet they appear to match them perfectly. Galileo's response was surrender he concluded that the notions of greater, lesser and equal simply cannot be applied to infinite quantities. The enigma he had entertained, was suggested unanswerable.

This remained the general consensus for the following two centuries. Infinity was acknowledged but left unexamined, a concept to deceitful for rigorous treatment. What was required was a new definition of 'size', one that could withstand infinite collections without contradiction.

Bijections and cardinality



Cantor's crucial insight was to define the size of a set (its cardinality), entirely in terms of correspondence rather than counting. Two sets A and B are said to have the same cardinality given there exists a bijection between them. This is a pairing in which every element in set A maps onto exactly one element in set B and vice versa. On either side, no element is left unpaired and no element appears multiple times. For finite sets this reinforces the typical notion of 'same number of elements', however for infinite sets, it provides, for the first time, a meaningful and consistent basis on which infinite sets can be compared.

With this definition, Galileo's paradox dissolves into an insight instead of a contradiction. The natural numbers $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ and the integers $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, have the same cardinality, even though $\mathbb{Z}$ is partly composed of negative numbers that $\mathbb{N}$ doesnt possess. The bijection is explicit

$$0 \leftrightarrow 0 \,,\ 1 \leftrightarrow 1 \,,\ -1 \leftrightarrow 2 \,,\ 2 \leftrightarrow 3 \,,\ -2 \leftrightarrow 4 \,,\ \ldots$$

Every integer appears exactly once on the left and every natural number exactly once on the right once on the right

$$\therefore\ |\mathbb{N}| = |\mathbb{Z}|$$

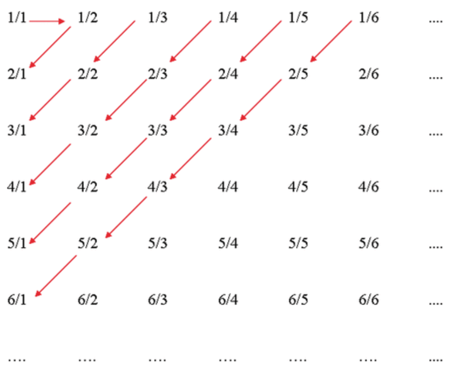
In order to interpret this strange arithmetic intuitively, consider the famous Hilbert's Hotel, a through experiment conjured up by David Hilbert. Imagine a hotel with infinite rooms 1, 2, 3, ... and so on, all of which are occupied. A new guest arrives, and in a finite hotel, they are turned away as this is impossible.

Instead, the manager simply asks everyone to move up one room so guests in room N move to room N+1. As a result, room 1 is now free for the newcomer. The Hotel was previously full, a guest was added, and it was filled up once again. The capacity of the hotel is not diminished or restored by having or not having guests. This is not a paradox, rather it is simply what infinite cardinality means.

Countably infinite sets

A set is known as countably infinite if a bijection exists between it and the natural numbers – meaning its elements can (in principle) be listed in a sequence: a first element, a second, a third, a fourth, and so on without end.

It is perhaps surprising that all rational numbers $\mathbb{Q}$ - all fractions p/q where p and q are integers and q ≠ 0 – are all also countably infinite. The rational numbers are incredibly dense as between any two numbers exists and infinite number of rational numbers, and they seem to fill the number line more thoroughly than integers do. Nevertheless, Cantor showed they are the same size of $\mathbb{N}$



The proof is elegant using a ‘zigzag’ enumeration. Arrange all positive fractions in an infinite grid, with denominators along the top, and numerators down the side. Beginning at 1/1 trace a diagonal path, moving right to ½ then southwest to 2/1, then across two the next column, moving down and left diagonally once again – visiting every cell exactly once. This path lists every positive rational in a sequence including negatives and zeroes, by the same interleaving trick as before, all of $\mathbb{Q}$ is countable.

The cardinality that is shared between $\mathbb{N}, \mathbb{Z}, and\ \mathbb{Q}$ is denoted by $\aleph_0$- the first and smallest infinite cardinal. This raises the decisive question - ‘are all infinite sets countable?’. If the answer is yes, there would only be one size of infinity. Georg cantor proved the answer is no.

Cantor’s diagonal argument

The following proof is amongst the most celebrated in mathematics. It is elementary in its tools - requiring only decimal expansions and a pinch of logical cunning – yet nevertheless, its conclusion is one of the most surprising in history

The claim: the set of real numbers $\mathbb{R}$ are not countably infinite.

Proof: suppose the $f : \mathbb{N} \longrightarrow [0, 1]$ is any function. Make a table of values of *f,* where the first row contains the decimal expansion of $f(1)$, the second row contain the decimal expansion of $f(2)$, the nth row contains the decimal expansion of $f(n)$. Perhaps *f(1)* could be $\pi/10$, $f(2)$ could be 37/99, $f(3) = 1/7$ and so on creating a table that begins to look like this:

n	$f(n)$												
1	0	.	3	1	4	1	5	9	2	6	5	3	...
2	0	.	3	7	3	7	3	7	3	7	3	7	...
3	0	.	1	4	2	8	5	7	1	4	2	8	...
4	0	.	7	0	7	1	0	6	7	8	1	1	...
5	0	.	3	7	5	0	0	0	0	0	0	0	...
$\vdots$	$\vdots$												

Can every number in the [0, 1] appear somewhere in the table?

The answer is no, as there are lots of numbers that can't appear.

For example, take the leading row of diagonals and highlight them, creating a table that looks as such:

n	$f(n)$												
1	0	.	**3**	1	4	1	5	9	2	6	5	3	...
2	0	.	3	**7**	3	7	3	7	3	7	3	7	...
3	0	.	1	4	**2**	8	5	7	1	4	2	8	...
4	0	.	7	0	7	**1**	0	6	7	8	1	1	...
5	0	.	3	7	5	0	**0**	0	0	0	0	0	...
⋮	⋮												

The highlighted digits are 0.37210….

Now add 1 to each of these digits in order to get 0.48321

This number can't appear in the table as;

It differs from *f(1)*, in the first digit

It differs from *f(2)*, in the second digit

It differs from *f(n)*, in the nth digit

So, it can't = f(n), for any value of n – thus it can't appear in the table

And as it turns out, many unrepresentable real numbers can't be represented, we could subtract 1 from each digit of the leading diagonal, we could subtract 3 from even numbered digits and add 7 to odd numbered digits.

What is the point of all this? It reveals that the function *f* can't possibly be onto – there will always be, infinitely many, missing values. Therefore, a bijection does not exist between ℕ and [0, 1]. So there exists different cardinalities.

Power sets and the infinite hierarchy

The existence of two distinct, infinite cardinalities - $\aleph_0$ and $\mathfrak{c}$ - might suggest that there are exactly two sizes of infinity. In fact, there are infinitely many, and Georg Cantor's most general theorem denotes why.

The power set of a set A, written as $\mathcal{P}$(A), is the set of all subsets of A. if A were = {1, 2, 3}, then $\mathcal{P}$(A) has 2^3= 8 elements. For any finite size of n, the number of elements is 2^n. Cantor proved that their relationship extends to infinite sets

Cantors' theorem: For any set A – finite or infinite - $|\mathcal{P}(A)| > |A|$.

Proof: let $f : A \longrightarrow \mathcal{P}(A)$ be any function and define

$$X = \{a \in A \mid a \notin f(a)\}$$

For example, if A = {1, 2, 3, 4}, then perhaps $f(1) = \{1, 3\}$, $f(2) = \{1, 3, 4\}$, $f(3) = \{\}$, and $f(4) = \{2, 4\}$

In this case X does not contain 1 as $1 \in f(1)$, however, X does contain 2 as $2 \notin f(2)$ and 3 as $3 \notin f(3)$, and X does not contain 4 as $4 \in f(4)$, so X = {2, 3}.

So, can X = f(a) for some $a \in A$? If so, then either a belongs to X or it doesn't. However, by the definition of X, if *a* belongs to X then it doesn't belong to X and vice versa, which creates an impossible scenario – so X cannot equal f(a) for any *a*. thus, similarly to the diagonal argument, this proves that f cannot be onto.

For example, the set $\mathcal{P}(\mathbb{N})$ - whose elements are sets of positive integers – has more elements than $\mathbb{N}$ itself; that is, it is not countably infinite.

As a consequence of this result, the sequence of infinite sets

$$\mathbb{N}, \mathcal{P}(\mathbb{N}), \mathcal{P}(\mathcal{P}(\mathbb{N})), \ldots$$

Must keep increasing in its cardinality. Thus, there are infinitely many different sizes of infinity.

The continuum hypothesis – an open question

Cantor himself worried about the gap between $\aleph_0$ and $\mathfrak{c}$. Due to this, he conjured up the continuum hypothesis, which is that there is no cardinality strictly between $\aleph_0$ and $\mathfrak{c}$, in other words, $\mathfrak{c} = \aleph_1$ where $\aleph_1$ is the next infinite cardinal after $\aleph_0$ in the hierarchy. Cantor spent years trying to either prove or disprove this, yet he never succeeded. Instead, it became the first of the Hilbert's 23 famous problems, presented in 1900 as the most important of open questions in mathematics.

When the resolution finally came, it was stranger than anyone had anticipated. In 1940, Kurt Gödel showed that the continuum hypothesis cannot be disproved from the standard axioms of set theory (ZFC – Zermelo-Fraenkel set theory with the axiom of choice), yet surprisingly in 1963, Paul Cohen showed, using a new theory of *forcing, that* the continuum hypothesis also cannot be proved from those same axioms. The continuum hypothesis if therefore formally independent of ZFC: it is consistent to assume it is true, and equally consistent to assume it is false. The standard foundations of maths are simply silent on the matter.

This is not a failure of mathematics – instead it's one of its deepest discoveries. Similarly to how Euclid's parallel postulate turned out to be independent of his other axioms, giving rise to non-Euclidean geometry, the continuum hypothesis reveals that the landscape of infinite cardinals is richer and more ambiguous than any fixed system of axioms can fully capture.

Conclusion

We began with a question that was almost nonsensical: can infinity be greater than infinity? The answer, established through rigorous mathematics, is yes - and the proof it both profound and accessible. Cantor's diagonal argument demonstrates that real number cannot be listed in any sequence making $\mathfrak{c}$ strictly greater than $\aleph_0$. His general theorem of power sets reveals that this is no isolated incident: there is an infinite ascending tower of distinct infinities, each unreachable from below by any bijection.

What makes Cantor's achievement remarkable is not merely the stunning results, but the method. By replacing vague geometric intuition with the precision of bijections and cardinality, he transformed infinity from a philosophical mystery to a mathematical object, that can be compared, studied, measured and proved things about.

However, the continuum hypothesis reminds us that even within this framework, fundamental questions remain open, as they transcend the reach of our axioms. In this sense, Cantor did not merely solve a problem about infinity but revealed that infinity is a vast enough subject to contain questions that mathematics itself cannot answer.

References

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