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Not Rocket Science

Introduction

This year has been a wonderful year for space enjoyers – with the Artemis II mission, and the Project Hail Mary movie recently coming out (although I'd hold the book is much better), I've seen more people come to appreciate space just over the past month – and in social media I've even seen a resurgence in the game Kerbal Space Program.

While I personally hold the belief that the work we do on Earth is much more important than gazing out at the stars, I can't help but feel a pull towards the incredible field of physics that is rocket science and astronomy – which leads me to try to understand how such things work properly.

The main example I will be focusing on in this essay is rockets and mass – a problem that comes up in almost every element of space travel, and every article I described above. Many common examples and questions assume that the mass of the rocket remains constant, when I know that is off by miles – most of a rocket's mass is fuel, realistically - and that is what I am attempting to model and find out here: does the difference in mass by lost fuel really matter that much?

The Problem

To address the idea of the major difference in how the problem is presented with and without the consideration of a changing mass, we will address both models directly.

Let's imagine a rocket leaving Earth: to escape the Earth's gravitational pull , its kinetic energy needs to be greater than or equal to the potential energy that holds it to the ground, or:

$$\frac{1}{2}mv^2 \geq \frac{GMm}{r}$$

(where m is mass of rocket, M is mass of Earth, v is escape velocity, r is radius of Earth and G is the gravitational constant)

Now this easily rearranges to the equation on the right, which only contains constant values, so the actual velocity we need to escape earth is very easy to calculate, sitting at 11,181m/s to the nearest m/s (or approx. 25000mph, or over 100 times the max speed of an F1 car). Now that a goal has been set, we need to find out how to reach that goal effectively.

$$V \geq \sqrt{\frac{2GM}{r}}$$

For clarity, we should also state the original values of the rocket. Let's assume that our rocket has a mass of X kg, a payload of 10,000kg, a fuel mass of Y kg and a velocity of propellant on exit of about 4,500m/s. In the solution section, we will try and use the two together to provide an answer, and then we will explain exactly how there is a contradiction.

In the standard model of mechanics, the rocket is considered to have a constant mass along many other assumptions (e.g. thrust being constant, ignore air resistance, etc.), which would provide a simple solution: to reach escape velocity, we use an impulse equation: where we take the velocity of the gas propellant and multiply it by the mass of the fuel, then divide by the average mass to get the change in velocity. From there we can find the mass of the fuel given.

However, this is not the case. With a model of a changing mass as fuel is lost, we should be thinking more realistically with this launch. We will assume air resistance/drag to be negligible, as most launches face straight upwards and can mostly ignore air resistance anyway. We will also tackle the question of a constant thrust, and whether that is true, or whether it even matters in the first place.

The issue of thrust can be solved easily – the overall thrust of the rocket is the sum of the momentum thrust and pressure thrust, where:

- Momentum thrust is m (rate of mass of fuel expended) times v (velocity of gas at exit
- Pressure thrust is $(p_e - p_a)A$
    - p_e is the pressure of the gas at the exit
    - p_a is the atmospheric pressure
    - A is the area of the exit

Therefore, most of the aspects of thrust can be considered constant – apart from the atmospheric pressure, which will lower as the rocket gets higher. This means that we can model the thrust of the rocket as T - Up_a - this isn't particularly significant, though, as it just shows the force of thrust will increase as the atmospheric pressure decreases – this is just a fun fact I wanted to include.

To solve the issue of the mass changing, we can use Tsiolkovsky's equation – let's assume our rocket of mass M is moving at a velocity at a given moment v. At the next moment, the rocket uses some fuel of mass (dm) at a speed (v_e) to increase velocity (by dv). Let's portray these as two separate moments of momentum:

Initial: Mv

Final: $(M-dm)(v + dv) + dm(v - v_e) = Mv + Mdv - dmv_e - dmdv$ (neglect the higher order term dmdv, since it's tiny)

Since momentum is conserved, we can equalise this to get:

$Mv = Mv + Mdv - dmv_e$

or $Mdv = v_e dm$

To rearrange this into a differential:

$dv = v_e / M \; dm$

Now we can integrate both sides, with our limits of integration being the start and end point of the burn of fuel, to have the change in velocity be equal to $-v_e$ times $(\ln(M_i) - \ln(M_F))$, or $-v_e \ln(M_i/M_F)$.

That was a lot of work, and I can see that it may seem meaningless if you weren't paying close attention, but we now are going to bring this right back to the beginning – by making the change in velocity equal to the velocity needed to escape, we can see a new equation forming:

$$V_{esc} = V_e \ln \frac{M_i}{M_F}$$

(vesc is escape velocity, all others are the same)

Since the escape velocity for Earth is around 11,181 m/s, we need the product of the velocity of fuel expulsion and the natural log of the ratio of initial mass:final mass to equal or exceed that number.

To rearrange for an exponential equation:

$$\frac{m_i}{m_F} = e^{\frac{v_{esc}}{v_e}}$$

Now, to make things simpler, we'll assume our rocket has the maximum fuel expulsion velocity for liquid propellant, which I googled to find a value of 4500 m/s. To plug this into the equation, we must do 11181 / 4500, which is approximately 2.48, and then e^2.48, which is approximately 12.

What this translates to is this: for a rocket with the maximum fuel expulsion velocity for liquid propellant, the mass of fuel must be 12 times the mass of the payload that the rocket is carrying to space.

The Solution

Obviously, from the work carried out before these models already contradict – but just to make it obvious to all readers, lets carry the equations with the values from before:

Rocket values:

- Initial mass: X kg

- Fuel mass: Y kg

- Velocity of propellant: 4,500m/s

- Payload mass: 10,000kg

Using these numbers to get us an initial mass, let's use Tsiolkovsky's equation in its exponential format:

X = 10,000 x e ^ (11181/4500)

= 119,971.2kg (or 120,000 kg for a nice approximation)

If the rocket truly never got lighter and stayed at a constant weight, the former model would predict this amount of fuel:

Y = (11181 x 120000) / 4500

= 298,160kg

However, the issue here is obvious. The initial mass (from Tsiolkovsky's) includes the initial fuel mass; however, the fuel mass is larger than the initial mass? Obviously one of these equations is wrong.

We can find the answer to that question just by looking at the format of each equation. Tsiolkovsky's model understands that fuel is disposed of during flight, and to a considerable extent – showing a logarithmic solution that provides a convergence of mass to a point. The constant mass model shows the opposite of that, a linear function where the mass will diverge away from a certain point.

Conclusion

Now that we've gone through solving each models to their end points regarding escaping the earth, we can solve the question proposed in the introduction: through comparing a constant mass model and Tsiolkovsky model, we showed that the Tsiolkovsky model converges onto a mass point, and with our first works we also showed that the amount of fuel needed to achieve escape velocity was much larger than the payload. We also showed that with the idea of mass being lost, a constant mass model simply doesn't work.

Therefore, the Tsiolkovsky model works way better than the constant mass model in finding the mass required for exiting the Earth.