kayak_{36} is a Palindromic Number
or
On Palindromic Numbers of Differing Bases

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Contents

0.1 Introduction

> **Palindromic**: A term [used] to describe a word, a number, a phrase, or other sequence of symbols that read the same backwards as they read forwards.
> - https://en.wikipedia.org/wiki/Palindrome

A palindromic number is a number that works the same way as a word palindrome does[1]. And, unlike palindromic words, there are an infinite amount of palindromic numbers in any given base[2]. This can easily be shown by writing the number 11 in a given base B, and placing "0" in the center, defined by the sequence S as follows:

$$S = \{11_B, 101_B, 1001_B, 10001_B, \ldots, B^n + 1\}$$

Theorem 1

For any given base B, there are n palindromes with $x \geq 1$ digits, where n is defined in base-10 as follows:

$$N = (B-1) \cdot B^{\left\lfloor \frac{x-1}{2} \right\rfloor} \tag{1}$$

So, by Theorem 1, there are 32_{10} 11-digit base-2 and 900_{10} 3-digit base-10 palindromes. But, for a given base B, how do we find the set of all palindromes[3] P_B?

[1] For example, 373_{10} is a base-10 palindromic number
[2] For the purposes of this essay, "any given base" refers to a number written in any base $B \in \{n \in \mathbb{Z} | n \geq 2_{10}\}$
[3] For the purposes of this essay, I will only be discussing positive palindromic numbers

Chapter 1

Constructing Palindromic Numbers

1.1 Brute Force Algorithm

Before we look at how to generate palindromic numbers, we should look for an algorithm that lets us determine if a number is palindromic or not.
To do this, we need to define a function $R(n)$ that reverses a number n written in base B that is K digits long[1]:

$$R(n) = \sum_{i=1}^{K} \left\lfloor \frac{n \mod B^{K+1-i}}{B^{K-i}} \right\rfloor \cdot B^{i-1} \tag{1.1}$$

1) $R(n)$ is n reversed
2) If $n = R(n)$ (or $n - R(n) = 0$), then n is a palindrome.

Here's a base-2 example with $n \in [1_2, 1001_2]$(palindromes highlighted in green):

n	$R(n)$	$n - R(n)$
1_2	1_2	0
10_2	1_2	1_2
11_2	11_2	0
100_2	1_2	11_2
101_2	101_2	0
110_2	11_2	11_2
111_2	111_2	0
1000_2	1_2	111_2
1001_2	1001_2	0

Although this method is guaranteed to work, it is highly inefficient, an issue that is fixed in the following two algorithms.

[1]For a positive integer N written in base B, $K = \left\lfloor \frac{\ln(n)}{\ln(B)} \right\rfloor + 1$

1.2 Mirror Method

To generate palindromes from a given non-negative integer $n \in \{x \in \mathbb{Z} | x \geq 0\}$:

Algorithmic Approach

1) Let n be a non-negative integer written in base B.
2) Let n' be the reverse of n (e.g., $n = 20_3 \rightarrow n' = 02_3$) without removing leading zeroes.
3) Let D be the set of all possible digits for a given base B, or $D \in [0, B) \cap \mathbb{Z}$
4) For each digit $d \in D$:
5.1) Place d between n and n'. The resulting number will be an "odd palindrome" (palindromic number with an odd amount of digits).
5.2) Place d twice between n and n'. The resulting number will be an "even palindrome" (palindromic number with an even amount of digits).

Alternatively, using the function defined in equation 1.1, $R(n)$, we can create two new functions to generate palindromes without having to use an algorithm. Given a number $c \in D$ and a positive integer n written in base B that is K digits long:

$$M_O(n, c) = n \cdot B^{K+1} + c \cdot B^K + R(n) \tag{1.2}$$

$$M_E(n, c) = n \cdot B^{K+2} + 11_B \cdot c \cdot B^K + R(n) \tag{1.3}$$

We can use these two functions to generate almost every palindrome in a given base. Here's a base-3 example with $n \in [1_3, 100_3] \cap \mathbb{Z}$

n	n'	d	$n+d+n'$	$n+d+d+n'$	$M_O(n)$	$M_E(n)$
0_3	0_3	0_3	0_3	0_3	DNE	DNE
0_3	0_3	1_3	1_3	11_3	DNE	DNE
0_3	0_3	2_3	2_3	22_3	DNE	DNE
1_3	1_3	0_3	101_3	1001_3	101_3	1001_3
1_3	1_3	1_3	111_3	1111_3	111_3	1111_3
1_3	1_3	2_3	121_3	1221_3	121_3	1221_3
2_3	2_3	0_3	202_3	2002_3	202_3	2002_3
2_3	2_3	1_3	212_3	2112_3	212_3	2112_3
2_3	2_3	2_3	222_3	2222_3	222_3	2222_3
10_3	01_3	0_3	10001_3	100001_3	10001_3	100001_3
10_3	01_3	1_3	10101_3	101101_3	10101_3	101101_3
10_3	01_3	2_3	10201_3	102201_3	10201_3	102201_3
11_3	11_3	0_3	11011_3	110011_3	11011_3	110011_3
11_3	11_3	1_3	11111_3	111111_3	11111_3	111111_3
11_3	11_3	2_3	11211_3	112211_3	11211_3	112211_3
20_3	02_3	0_3	20002_3	200002_3	20002_3	200002_3
20_3	02_3	1_3	20102_3	201102_3	20102_3	201102_3
20_3	02_3	2_3	20202_3	202202_3	20202_3	202202_3
21_3	12_3	0_3	21012_3	210012_3	21012_3	210012_3
21_3	12_3	1_3	21112_3	211112_3	21112_3	211112_3
21_3	12_3	2_3	21212_3	212212_3	21212_3	212212_3
22_3	22_3	0_3	22022_3	220022_3	22022_3	220022_3
22_3	22_3	1_3	22122_3	221122_3	22122_3	221122_3
22_3	22_3	2_3	22222_3	222222_3	22222_3	222222_3
100_3	001_3	0_3	1001001_3	10011001_3	1001001_3	10011001_3
100_3	001_3	1_3	1002001_3	10022001_3	1002001_3	10022001_3
100_3	001_3	2_3	1003001_3	10033001_3	1003001_3	10033001_3

Unfortunately, the function-based approach does not provide us with the trivial palindromes included in set D. However, to construct the set of all palindromes in a given base B, P_B, we can simply join the set of all palindromes generated by M_O and M_E with the set D.

For a given base B, the set of all non-negative palindromes P_B is defined as follows:
The set of all possible digits for is defined by $D = \{x \in \mathbb{Z} | 0 \leq x < B\}$
The set of all odd palindromes that are $\geq B$ is defined by $P_{B_O} = \{x | x = M_O(n, c) \land \exists n \in \mathbb{N} \land \exists c \in D\}$
The set of all even palindromes that are $\geq B$ is defined by $P_{B_E} = \{x | x = M_E(n, c) \land \exists n \in \mathbb{N} \land \exists c \in D\}$
Therefore $P_B = D \cup P_{B_O} \cup P_{B_E}$

For a given base B, there are $(B-1)B^{y-1}$ numbers that can be written using y digits[2]. Based on the method described above, the center number c placed between n and n' can assume B different values. So, by the Rule of Product, there are $N = (B-1) \cdot B^{y-1} \cdot B = (B-1)B^y$ possible combinations of the y-digit number n, and c. Since n' is dependent on n, it does not affect the total amount of possible palindromes for a given n with y digits. Furthermore, the mirror method will produce a palindrome that has either $2y+1$ or $2y+2$ digits. If we let $x = 2y+1$, then we will produce a palindrome with either x or $x+1$ digits. $x = 2y+1 \implies y = \left\{\dfrac{x-1}{2}, \dfrac{x}{2}\right\}$ therefore $N = (B-1) \cdot B^{\lfloor \frac{x-1}{2} \rfloor}$, which is exactly what Theorem 1 posited.

[2]For example, 3 digits in base-10 allows you to write every integer between 100 and 999 inclusive

1.3 Recursive Shell Method

The final method is a more visual approach to creating palindromes. To generate odd palindromes in a given base B, we begin by creating a grid with 1 column and $B+1$ rows. We will be looking at examples in base-3 for this section. An example of the base-3 base case is found on the next page:

This table will be called $P_{B_O}(0)$, the base case for our recursive method for finding palindromic numbers. Then, we will define our shell for each $P_{B_O}(n)$ with $n > 0$ as being a grid with $(B-1) \cdot B^n$ rows and $2n+1$ columns. To fill each shell, the algorithm below will be used:

1) Split the grid into $B-1$ groups with equal amounts of rows.
2) Fill the first and last column of each group with the same number m, with each group having a different digit m from the set $D = \{x \in \mathbb{N} | x < B\}$ which contains all possible non-negative digits in a given base B
3) Fill all remaining cells in the gird with 0

Example shells for $P_{3_O}(1)$ and $P_{3_O}(2)$ can be found on the next page. Then, the algorithm to assemble $P_{B_O}(n)$ continues as follows

4) For each group created in step 1, place $P_{B_O}(n-2)$ at the top, and $P_{B_O}(n-1)$ at the bottom

An example can be seen on the following pages. To construct the even palindromes, a second copy of the n-th row must be inserted in the center

All numbers on this page are written in ternary (Base-3)

Base case for odd palindromes in Base-3

0
1
2

Shell for $P_{3_O}(1)$

1	0	1
1	0	1
1	0	1
2	0	2
2	0	2
2	0	2

Shell for $P_{3_O}(2)$

1	0	0	0	1
1	0	0	0	1
1	0	0	0	1
1	0	0	0	1
1	0	0	0	1
1	0	0	0	1
1	0	0	0	1
1	0	0	0	1
1	0	0	0	1
2	0	0	0	2
2	0	0	0	2
2	0	0	0	2
2	0	0	0	2
2	0	0	0	2
2	0	0	0	2
2	0	0	0	2
2	0	0	0	2
2	0	0	0	2

$P_{3_O}(0)$

0
1
2

$P_{3_O}(1)$

1	0	1
1	1	1
1	2	1
2	0	2
2	1	2
2	2	2

$P_{3_O}(2)$

1	0	0	0	1
1	0	1	0	1
1	0	2	0	1
1	1	0	1	1
1	1	1	1	1
1	1	2	1	1
1	2	0	2	1
1	2	1	2	1
1	2	2	2	1
2	0	0	0	2
2	0	1	0	2
2	0	2	0	2
2	1	0	1	2
2	1	1	1	2
2	1	2	1	2
2	2	0	2	2
2	2	1	2	2
2	2	2	2	2

Chapter 2

Observations

2.1 Triangular Palindromes

The triangular numbers are the second diagonal of Pascal's triangle. The n-th triangular number is given by the equation $T(n) = \binom{n+1}{2} = \frac{(n+1)!}{2(n-1)!}$. Although it may seem like the triangular numbers have nothing to do with the palindromic numbers, an interesting pattern emerges when viewing their values in different number bases.

B	$T(2)$	$T(4)$	$T(6)$	$T(8)$	$T(10)$	$T(12)$	$T(14)$	$T(16)$	$T(18)$	$T(20)$	$T(22)$
2	11	1010	10101	100100	110111	1001110	1101001	10001000	10101011	11010010	11111101
3	10	101	210	1100	2001	2220	10220	12001	20100	21210	100101
4	3	22	111	210	313	1032	1221	2020	2223	3102	3331
5	3	20	41	121	210	303	410	1021	1141	1320	2003
6	3	14	33	100	131	210	253	344	443	550	1101
7	3	13	30	51	106	141	210	253	333	420	511
8	3	12	25	44	67	116	151	210	253	322	375
9	3	11	23	40	61	86	126	161	210	253	311
10	3	10	21	36	55	78	105	136	171	210	253
11	3	a	1a	33	50	71	96	114	146	181	210
12	3	a	19	30	47	66	89	b4	123	156	191

Based on this table, it seems that in a given base $B \geq 2$, the i-th triangular number such that $i = 2 \cdot (B-1)$ is a palindromic number that both starts and ends with 1, and has $B-3$ as its center digit. In other words, $T(i)$ is a palindromic number of the form $T_P(B) = B^2 + (B-3) \cdot B + 1$. From the table, this pattern holds true for for bases $B \in [2, 12]$, but how do we know if it holds true in any given number base?

To verify that this pattern is true for all bases $B \in [2, \infty)$, we need to some algebraic manipulation with the function $T(n)$ involving the factorial. Since $n! = n \cdot (n-1) \cdot (n-2)!$ and $T(n) = \frac{(n+1)!}{2(n-1)!}$, $T(n)$ can be rewritten as $T(n) = \frac{(n+1)!}{2(n-1)!} = \frac{(n+1)(n)}{2} \cdot \cancelto{1}{\frac{(n-1)!}{(n-1)!}} = \frac{n^2+n}{2}$. Plugging in $n = i = 2 \cdot (B-1)$, we get $T(i) = \frac{2^2(B-1)^2 + 2(B-1)}{2} = 2(B-1)^2 + B - 1 = 2B^2 - 4B + 2 + B - 1 = 2B^2 - 3B + 1$. If we expand $T_P(B) = B^2 + (B-3) \cdot B + 1$, we get $T_P(B) = B^2 + B^2 - 3B + 1 = 2B^2 - 3B + 1$, which is equal to $T(i)$, therefore $T_P(B) = T\left(2 \cdot (B-1)\right) = B^2 + (B-3)B + 1$. Based on this equation, the coefficient of the center term $B - 3 \geq 0 \Leftrightarrow B \geq 3$, so why does this pattern still work in base-2 where the center coefficient would be -1? Based on the definition of $T_P(B)$, $T_P(2) = 2^2 - 2 + 1$. Simplifying this equation yields $T_P(2) = 3_{10} = 11_2$, which follows the pattern of starting and ending with 1, but it is not a 3 digit number like the rest of the triangular numbers of this form.

2.2 Repdigit Squares

A repdigit is a number in a given base B with digits that are only the number 1. Repdigits, by definition are palindromic, but so are the squares of the first $B-2$ repdigits. Squaring a repdigit with K digits forms a palindrome with digits that increment by 1 until hitting K, and then decrease back to 1. The first 8 repdigits in base-10 form the squares below

n	T_n	$(T_n)^2$
1	11	121
2	111	12321
3	1111	1234321
4	11111	123454321
5	111111	12345654321
6	1111111	1234567654321
7	11111111	123456787654321
8	111111111	12345678987654321

2.3 No Palindrome With an Even Amount of Digits is Prime

Theorem 2

If P_B is the set of every palindromic number in a given base B, every number $n \in P_B$ with an even amount of digits K, then $(B+1)|n$, meaning n is divisible by $(B+1)$.

To prove this, we need to use modular arithmetic. Modular arithmetic involves taking the remainder b of a number a after it is divided by m, written as $a \equiv b \pmod{m}$.

If x is a nonnegative integer; a,b,c, and m are integers; and $a \equiv b \pmod{m}$, the following properties hold
1) $a^x \equiv b^x \pmod{m}$
2) $a - b = c \Leftrightarrow a \equiv c \pmod{m}$
3) $b|a$ iff $a \equiv 0 \pmod{b}$

Using these properties, we are then able to prove our original claim.

Proof:

1) Let $B \in \mathbb{N}$, $x \in \mathbb{N} \cup \{0\}$
2) $B - (B+1) = -1$, therefore $B \equiv -1 \pmod{B+1} \Leftrightarrow B^x \equiv (-1)^x \pmod{B+1}$
3) Since $B^x \equiv (-1)^x \pmod{B+1}$, $B^x = 1$ for all even x, and $B^x = -1$ for all odd x.
4) A number written in base B with $K \in [1, \infty)$ digits can be written as $\sum_{n=0}^{K-1} a_n \cdot B^n$. A number is palindromic iff $a_0 = a_{k-1}, a_1 = a_{k-2}, a_2 = a_{k-3}, \dots, a_n = a_{K-i-n}$, or $\sum_{n=0}^{K-1} a_n \cdot B^n = \sum_{n=0}^{K-1} a_{K-1-n} \cdot B^n$.
5) Let $n = a_0 + a_1B^1 + a_2B^2 + \cdots + a_{K-1}B^{K-1} \in P_B$
6) If K is even, then $n \equiv a_0 - a_1 + a_2 - \cdots - a_{K-1} \pmod{B+1}$
7) Let $c = a_0 - a_1 + a_2 - \cdots - a_{K-1}$
8) Since we are assuming $B+1|n$, Suppose $n \equiv c \equiv 0 \pmod{B+1}$. If $c = 0$, then $c \equiv 0 (\text{mod } B+1)$
9) c can be rewritten as $c = (a_0 - a_{K-1}) - (a_1 - a_{K-2}) + (a_2 - a_{K-3}) - + \cdots = \sum_{i=0}^{\frac{K}{2}-1} (-1)^i (a_i - a_{K-i-1})$
10) Since $n \in P_B$, $a_n = a_{K-i-n}$ and $a_n - a_{K-i-n} = 0$, therefore c must be equal to 0
11) $c = 0$ means $n \equiv c \equiv 0 \pmod{B+1}$, therefore $(B+1)|n$

Since every palindrome in base B with an even amount of digits must be divisible by $B+1$, they cannot be prime unless $B+1$ is prime, as $B+1$ would be a palindrome with an even amount of digits. Numbers like $11_2, 11_4, 11_6$ and 11_{10} are outliers to this rule of not being prime.

2.4 The Sum of the Reciprocals of the Palindromes

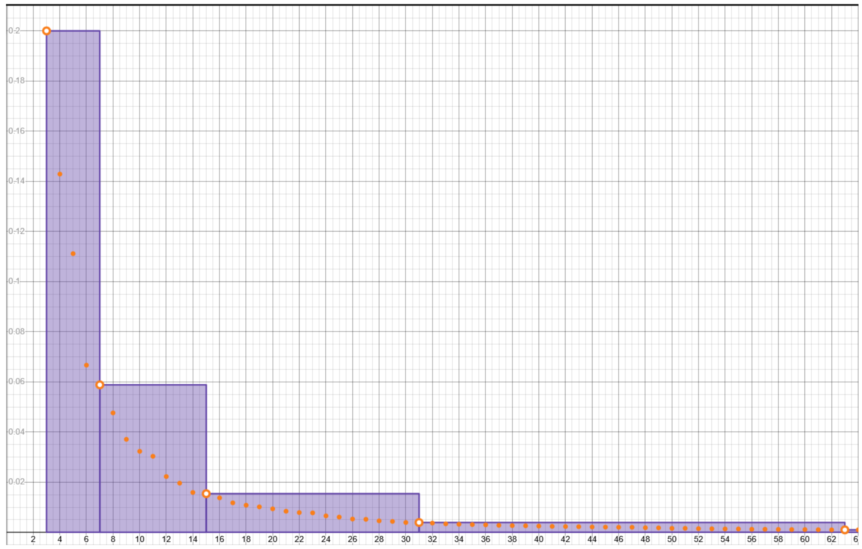
How do we know if the sum of the reciprocals of every non-zero palindrome in a given base converges to a finite value? Since we don't have a formula for the n-th palindrome in a given base B, we can try a geometric approach to proving whether or not this series diverges. Ideally, we could find the true sum by summing the areas of infinitely many rectangles with a width of 1 and a height of the reciprocal of the n-th palindrome. However, since this is not possible with what we know, we can attempt to overestimate the total area of these rectangles. If we overestimate the true sum of the series by using infinitely many rectangles that encompass more area than the original rectangles and the sum of these areas converges to a finite value, then the sum of the original rectangles must also converge to a lesser finite value.

To begin, we will look at what we already know. Based on the Mirror Method, we know that the smallest palindrome in a given base B with $K \geq 3$ digits is a member of the sequence $S = \{101_B, 1001_B, 10001_B, \ldots, B^n + 1\}$. Furthermore, we know based on Theorem 1 that the distance between these numbers is $(B-1)B$, $(B-1)B$, $(B-1)B^2$, $(B-1)B^2$, $(B-1)B^3$... and so on. Using this information, we can construct a table containing these values. The table below shows the j_i-th palindrome of the sequence containing all palindromes in a given base B, P.

i	j_i	P_{j_i}
1	$2 \cdot (B-1) + 1$	101_B
2	$j_1 + (B-1) \cdot B$	1001_B
3	$j_2 + (B-1) \cdot B$	10001_B
4	$j_3 + (B-1) \cdot B^2$	100001_B
5	$j_4 + (B-1) \cdot B^2$	1000001_B
6	$j_5 + (B-1) \cdot B^3$	1000001_B
$\vdots$	$\vdots$	$\vdots$
n	$j_{n-1} + (B-1) \cdot B^{\lfloor \frac{n}{2} \rfloor}$	$B^{n+1} + 1$

Since these are the lowest value palindromes with K digits, their reciprocals will be the highest out of every palindrome with K digits. If we construct a rectangle with the height of these reciprocals and a width equal to the amount of palindromes with K digits, we will have created an overestimate of the total area. However, if we use every palindrome from the sequence S we will have to test if a recursive series converges, which is inconvenient. To make it simpler to tell if the series converges, we can attempt to get rid of the recursive nature by making the width of each rectangle equal to the distance between every other palindrome in the sequence S. Since the reciprocals will always be decreasing, the rectangles height will still be an overestimate.

Using every other palindrome allows us to create rectangles with width $2(B-1) \cdot B^n$ and height $\frac{1}{B^{2n}+1}$. The image below shows this using the binary palindromes with our overestimate heights represented by the hollow orange circles.



Now that we have the dimensions of the rectangles we will use to overestimate the true sum, we can write their infinite sum as the series $A = \sum_{n=1}^{\infty} \frac{(B-1) \cdot B^n}{B^{2n}+1}$. To know definitively if this series converges, there are two tests that we must do. The first, the Limit Comparison Test (LCT), will help us compare our series A with another series A' to prove the convergence of both. The LCT works as follows:

Limit Comparison Test

Let S be the series $S = \sum_{n=0}^{\infty} a_n$ Let S' be the series $S' = \sum_{n=0}^{\infty} b_n$ If $\lim_{n\to\infty} \frac{a_n}{b_n} = c$ such that $c \in (0, \infty)$, and S' converges to a finite value, then S must also converge to a finite value.

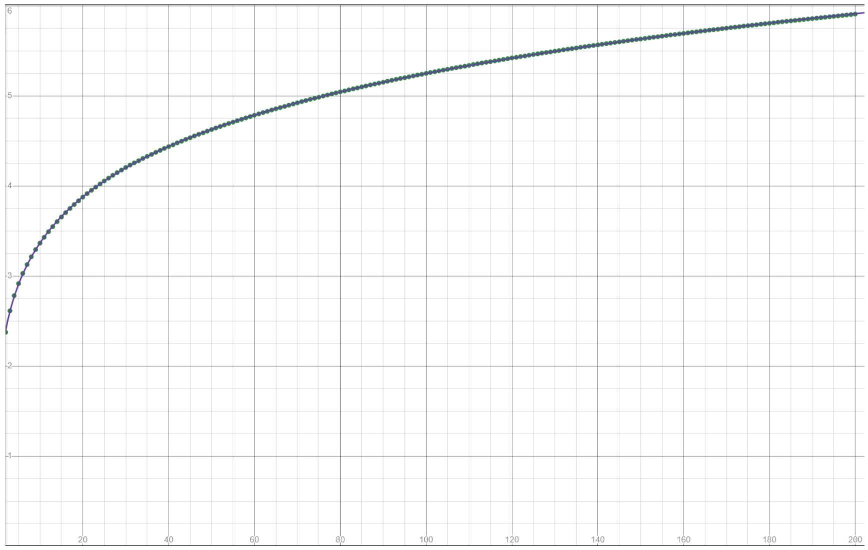
The second test we need to use is the Geometric Series Test (GST). The GST will allow us to prove the convergence of A' as follows:

Geometric Series Test

Given $a, r \in \mathbb{R}$, the series $S = \sum_{n=0}^{\infty} a \cdot r^n$ will converge iff $|r| < 1$

Using these two tests, we will set $a_n = \frac{(B-1) \cdot B^n}{B^{2n}+1}$, $A' = \sum_{n=1}^{\infty} \frac{(B-1) \cdot B^n}{B^{2n}} = \sum_{n=1}^{\infty} \frac{(B-1)}{B^n}$, and $b_n = \frac{(B-1) \cdot B^n}{B^{2n}}$. Then, $c = \lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{B^{2n}}{B^{2n}+1} \cdot \cancelto{1}{\frac{(B-1) \cdot B^n}{(B-1) \cdot B^n}}$. Multiplying the numerator and denominator by B^{-2n} gives us $c = \lim_{n\to\infty} \frac{1}{1+\frac{1}{B^{2n}}} = \frac{1}{1+0} = 1 \in (0, \infty)$, therefore by the LCT A and A' will either converge to a finite value or diverge. To check if both of them converge, we can now use the GST on A'. For the series A', $|r| = \frac{1}{B} < 1$, so by the GST A' must converge, and by the LCT, A must also converge. Since A, our overestimate, converged to a finite value, then the sum of the reciprocal of every palindrome must also converge to a finite value.

We can estimate this value by summing the reciprocals of the first 1000 palindromic numbers. Plotting these values in bases $B \in [2, 200]$ shows that the sums of the reciprocals of palindromes in differing bases lie on a radical function shown on the graph below, where $y \approx 6.4\,(x - 0.7)^{\frac{1}{10.6}} - 4.6$.



Chapter 3

Conclusion

Base 36 uses the digits 0-10 and the whole English alphabet. Since every letter is used as a digit, words that are palindromic in English are able to be written in base 36 as a number palindrome, like kayak$_3$6. Neat, huh?

Hopefully you had as much fun learning these facts about palindromic numbers as I did discovering the things I wrote about in this essay. On the next page are some graphics to visualize the first few palindromes in base 2, 3, 4, and 5. Hopefully you see the patterns I first saw that inspired me to write this whole paper.