The Mathematics of Origami:
How Paper Folding Solves the Unsolvable

1 Introduction: Three Problems That puzzled the ancients

Imagine being handed a challenge by the greatest minds of ancient Greece, a challenge so deceptively simple that generations of brilliant mathematicians would spend centuries attempting it, only to fail every single time. This challenge consisted of taking any angle and dividing it perfectly into three equal parts, using only a straightedge and compass. No protractors, no calculators. Just two tools and a piece of paper.

This problem called ‘angle trisection’ was one of three infamous puzzles that consumed Greek geometers. The other two were 'doubling the cube' (constructing a cube with exactly twice the volume of a given cube) and 'squaring the circle' (drawing a square with the same area as a given circle). These were not random curiosities. They represented the limits of what two simple tools could achieve, and for over two thousand years, no one could prove whether they were solvable or not. [1]

In 1837, the French mathematician Pierre Wantzel finally delivered the verdict: two of these problems — angle trisection and doubling the cube — are provably impossible using only a compass and straightedge. The key insight is that compass-and-straightedge constructions can only ever produce lengths built from square roots — but trisecting an angle required solving a cubic equation, which demands cube roots. The tools of Euclid simply cannot reach that far [2].

And yet, it turns out that a sheet of paper and a few well-chosen folds can do what two thousand years of Greek geometry could not.

2 A New Geometric Language

Origami, the Japanese art of paper folding might seem like an unlikely solution to this problem. But in 1986, the French mathematician Jacques Justin wrote down seven rules later formalised as the Huzita-Hatori axioms named after Humiaki Huzita, who rediscovered the first six, and Koshiro Hatori, who identified the seventh. They completely describe every possible single fold you can make on a piece of paper. Just as Euclid's geometry rests on five foundational hypotheses origami geometry rests on these seven axioms, each one specifying a different way of aligning points and lines through a crease [3].

The axioms are surprisingly intuitive. The first simply says: given two points P_1 and P_2, there is a unique fold that passes through both of them. This is the origami equivalent of drawing a straight line. Axiom 2 says: given two points P_1 and P_2, there is a fold that places P_1 exactly onto P_2 equivalent to finding a perpendicular bisector. The remaining axioms describe folding lines onto lines, dropping perpendiculars, and more complex alignments.

But Axiom 6 ,the most powerful of all states, given two points p_1 , p_2 and two lines l_1 , l_2 , there is a fold that simultaneously places p_1 onto l_1 and p_2 onto l_2 [3]. This single fold is the operation that changes everything. Robert Lang, one of the world's leading origami mathematicians, has proven that these seven axioms are complete there is no eighth type of single fold that they miss [4][10].

3 Why Origami Beats Euclid

To understand exactly why origami surpasses compass-and-straightedge geometry, we need to think in terms of algebra. Every compass and straightedge construction ultimately reduces to finding intersections of lines (described by linear equations) and circles (described by quadratic equations). A compass and straightedge can handle square roots but no matter how cleverly you combine them, they can never reach a cube root.

Trisecting an angle, however, fundamentally requires solving a cubic equation. We can see this concretely through the triple angle formula for cosine, which states that for any angle θ:

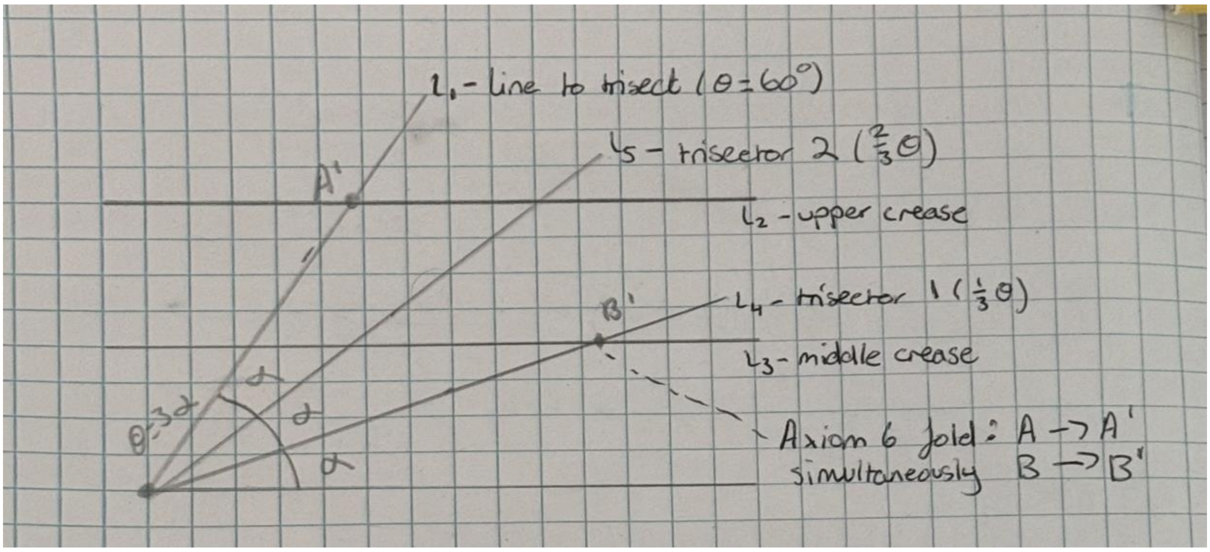
$$\cos(3\theta) = 4\cos^3(\theta) - 3\cos(\theta)$$

Now suppose we wish to trisect a 60° angle a seemingly simple case. We know that cos(60°) = 1/2. Setting c = cos(20°) (the value we need to construct), the formula gives:

$$4c^3 - 3c = 1/2$$

Rearranging:

$$8c^3 - 6c - 1 = 0$$



This cubic equation cannot be broken down any further using rational numbers it is, in mathematical terms, irreducible. That is what makes trisection impossible with Euclid's tools and compass and straightedge constructions can only reach quadratic equations and is limited by a cubic equation. [6]

Now considering Axiom 6 of origami. Geometrically, performing this fold is equivalent to finding a straight line that is simultaneously tangent to two different parabolas one with focus p_1 and fixed line l_1, and another with focus p_2 and fixed line l_2 [3]. From conics, finding a common tangent to two parabolas generally requires solving a cubic equation. The fold encodes cubic arithmetic in a physical manner. The mathematician Margherita Beloch realised this as far back as 1936, showing that origami could solve general cubic equations decades before the axioms were formally written down [5].

Origami geometry is in short, evidently more powerful than Euclidean geometry. With, compass and straightedge reach quadratic equations, whilst Axiom 6 reaches a cubic degree. The field of numbers constructable by origami is simply an algebraic extension of the field constructible by compass. [7]

4 Abe's Method and Its Proof

The most elegant construction for trisecting an arbitrary angle with origami was devised by the Japanese mathematician Hisashi Abe in 1980, and the proof of why it works relies only on properties of isosceles triangles nothing more advanced than GCSE level geometry. [8][9].

Suppose an acute angle θ sits at the lower-left corner B of a square sheet of paper, formed between the base edge and a fold line l_1. The construction proceeds as follows:

Step 1: Create a horizontal crease l_2 anywhere above the base.

Step 2: Fold the base up to meet l_2, creating crease l_3 exactly halfway between them. Let A be the left endpoint of l_2.

Step 3 (the Axiom 6 move): Fold so that A lands on l_1 (call the new position A') and simultaneously B lands on l_3 (call the new position B'). This step is the one impossible with Euclid's tools.

Step 4: Without unfolding, crease along l_3 to define a new fold l_4. Then unfold everything.

Step 5: Fold the base edge up to l_4, creating crease l_5.

The lines l_4 and l_5 now trisect the angle θ. But why? Here is the proof [8], adapted from Richeson (2012).

Let α = ∠DBE, where D is the point on l_4 at the base. We wish to show that θ = 3α. First, since l_3 was constructed as the midpoint between the base and l_2, the Axiom 6 fold ensures that AB = CD (where C is the image of A and D the image of B under the fold). This means ABDC is an isosceles trapezoid, and triangle ABD is isosceles with AB = CD.

Now, since BE and DF are parallel (they are both horizontal), alternate interior angles give:

$$\angle DBE = \angle BDF = \alpha$$

Because DF is the altitude of the isosceles triangle ABD (it bisects it by symmetry):

$$\angle BDF = \angle ADF = \alpha$$

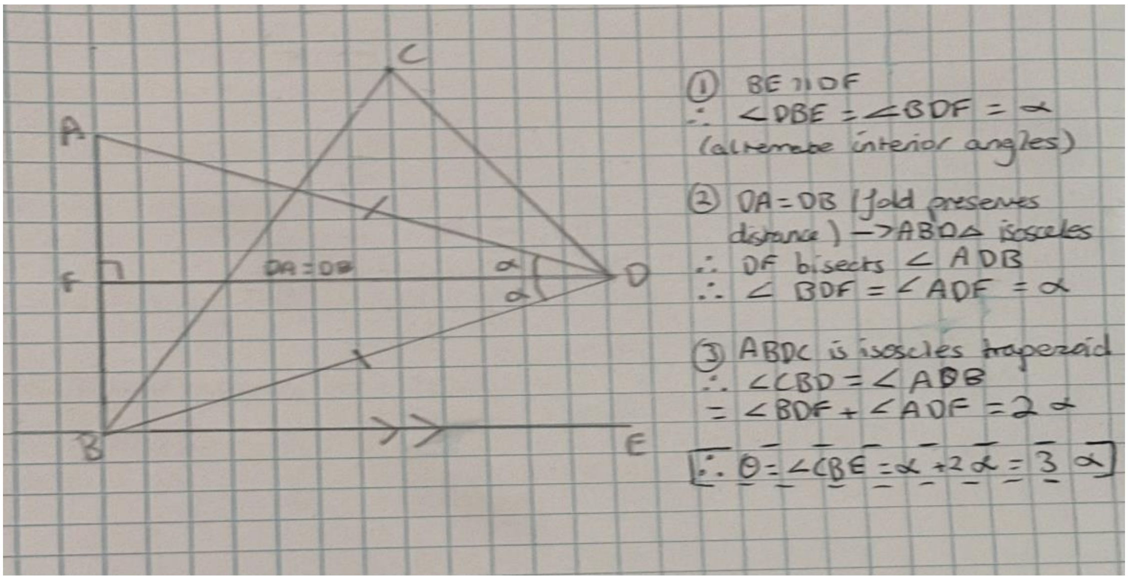
Since ABDC is an isosceles trapezoid, triangles ABD and BCD are congruent. Therefore:

$$\angle CBD = \angle ADB = \angle BDF + \angle ADF = \alpha + \alpha = 2\alpha$$

Putting it all together:

$$\theta = \angle CBE = \angle DBE + \angle CBD = \alpha + 2\alpha = 3\alpha$$

The angle is trisected exactly not approximately. Every step follows from congruent triangles and parallel lines, with one exception: the simultaneous two-point fold in Step 3. That single crease is the moment where the cubic equation gets solved not on a page with algebra, but physically, with your hands.



5 Origami in the real world

None of this is purely theoretical. However, The mathematics of origami has found its way into some of the most demanding engineering problems of our time.

For example, the challenge of fitting a solar panel array which might span the width of several buses when deployed inside a rocket barely five metres across. NASA engineers have solved this using origami mathematics. The International Space Station's solar arrays fold using a Z-pattern, while the Mars Phoenix lander used a fan fold called ‘Ultra Flex’, both designed using these mathematical principles. Robert Lang himself gave his expertise to NASA's ‘Star shade’ project, a 26 metre disc designed to block starlight so that telescopes can photograph orbiting exoplanets. This crease pattern which must fold reliably and compress tightly enough to fit inside a rocket was derived using computational origami [13].

Within vehicles, origami mathematics governs how your car's airbag is folded. Engineers used origami-based algorithms developed by Lang to produce the first geometrically correct airbag folding method one that could be simulated on a computer rather than tested in an expensive crash, resulting in fewer physical crash tests and a more reliably deploying airbag [12].

In medicine, foldable stents which are small mesh tubes used to prop open narrowed arteries are inserted into the body in a tightly folded form and then expanded at the target site using principles from the 'waterbomb base' fold. Such origami-inspired strategies are now being explored for tissue scaffolds, drug delivery devices, and surgical robotics [11]. The US National Science Foundation summarises the reach of this work simply: 'research using the science behind origami has produced breakthroughs in fields as diverse as architecture, medicine, biology, and robotics' [14].

6 Conclusion

It might sound strange about the fact that a piece of paper turns out to be a more powerful geometric instrument than the compass and straightedge that defined mathematical construction for thousands of years. The ancient Greeks were not wrong to use their tools; they simply could not know that a different set of axioms, formalised not until 1986, would unlock what they could not reach.

The mathematics of origami illustrates that the limits of the 'impossible' are always drawn with respect to a particular set of assumptions. Wantzel proved in 1837 that trisecting an angle requires solving an irreducible cubic equation and that compass and straightedge geometry is, algebraically, stuck at degree two. But a fold encoded in Axiom 6 reaches degree three quite easily. The same cubic arithmetic that makes trisection impossible with Euclid's tools makes it straightforward with a sheet of paper.

Any time you fold a piece of paper, even carelessly in half, you are performing an action that lies beyond the reach of ancient Greek geometry. You are, quite literally, doing something the greatest mathematicians of ancient times could not.

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