# SKIING THE BRACHISTOCHRONE: THE FASTEST WAY DOWN

**By James Moseley**

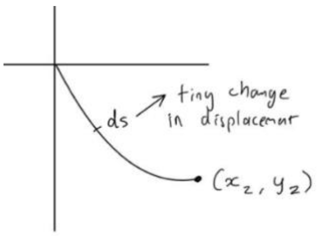
## INTRODUCTION

In 1696, a Swiss mathematician by the name Johann Bernoulli introduced a challenge to Europe "**Given two points A and B in a vertical plane, what is the curve traced out by a point acted on only by gravity, which starts at A and reaches B in the shortest time?"** This challenge intrigued many mathematicians such as Newton and Leibniz among others [1]. Many submitted but there was one standout. Out of all the entry's, there was one anonymous answer that caught Bernoullis attention in which he famously said, "I recognise the lion by his claw." This submission was, of course, was by Isaac Newton [2].

So how can this problem be utilised? Well, the proposed idea is key to many real-world applications such as roller coasters, optics, and such however what I will be relating it to, is skiing where finding the fastest time down is essential. We will start with the previously mentioned idea presented by Bernoulli however to change it towards the idea of skiing we will also include friction later as it would be more accurate to an actual skiing scenario without being too complex.

## THE CLASSICAL BRACHISTOCHRONE PROBLEM

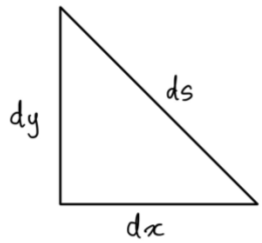
The classical brachistochrone problem mentioned before was attempted by mathematicians in a variety of ways such as Galileo incorrectly concluding the answer was a circular arc [1]. So how do you solve it? Now there are multiple ways to solve this problem but the way we will look at is using calculus of variations and the time functional.




$$t = \int \frac{ds}{v}$$

To start we have to think about how to express the minimum time. If we think about adding lots of small sections of this path, this can this be shown using this integral.

$$ds = \sqrt{dx^2 + dy^2}$$




Using Pythagoras's theorem, it can be assumed that a small change in displacement $ds$ along a curve can be shown by using a small horizontal change $dx$ and a small vertical change $dy$ as sides of a right-angled triangle. Therefore, to calculate $ds$ we would use Pythagoras and this would be the result.

$$ds = \sqrt{x'^2 + 1}\, dy$$

As we know that $x'dy$ is equal to $dx$ ,this can be used for substituting and factoring out $dy$ and so we arrive at this new formula for $ds$ which we can now substitute into our original integral

$$\frac{1}{2}mv^2 = mgy \qquad v = \sqrt{2gy}$$

To find velocity, we can use the idea of conservation of energy as all the potential energy will be converted to kinetic energy when going down the curve, giving us velocity in terms of acceleration from gravity and the height which also will be substituted into the integral.

$$t = \frac{1}{\sqrt{2gy}} \int_0^{y2} \sqrt{\frac{x'^2 + 1}{y}}$$

Now when we substitute in what we have for $ds$ and $v$ we get this integral. After taking out the constant term, this is what the integral is with $y2$ being the vertical component of our point at the bottom of our curve.

$$t = \int f(x, x', y)\, dy \qquad \frac{\partial f}{\partial x} - \frac{d}{dy}\left(\frac{\partial f}{\partial x'}\right) = 0$$

According to Euler-Lagrange's equation from Calculus of Variations, a curve which can be represented by the integral above has a stationary point that must satisfy the following equation [3]. In the case of the brachistochrone, this is a minimum point as we are finding the minimum time for the curve.

$$f = \sqrt{\frac{x'^2 + 1}{y}} \qquad \frac{\partial f}{\partial x} = 0$$

We set our function to be equal to the expression inside the integral. Looking at the first part of the equation, we must take the partial derivative of the function with respect to $x$, as there is no $x$ variable in this equation, the whole expression can be treated as a constant, and its derivative would therefore be 0

$$\frac{d}{dx}(\frac{\partial f}{\partial x'}) = 0$$

For the whole equation to be true and equal zero however, the derivative of the partial derivative of $f$ with respect to $x'$ must be 0 and therefore the partial derivative must be a constant which would therefore differentiate to 0 and make the expression true.

$$\frac{\partial f}{\partial x} = \frac{x'}{\sqrt{y(x'^2 + 1)}}$$

Taking the partial derivative of $f$ with respect to $x'$ gives us this result.

$$\frac{x'^2}{y(x'^2 + 1)} = \frac{1}{2a}$$

We can then use a simplification method in order to turn this expression into a constant where we square the expression to remove the square roots and setting it equal to the constant above and solving for $x'$, where $a$ is a constant. By doing this we "get rid" of the variable $x'$ and the expression can hence be treated as a constant.

$$x' = \sqrt{\frac{y}{2a - y}} \qquad x = \int \sqrt{\frac{y}{2a - y}}\, dy$$

Solving for $x'$, we cross multiply and collect terms before dividing and taking the square root in which we get this expression. We can then take the integral of the expression to find an expression for $x$.

$$y = a - a\cos\theta \qquad x = \int \sqrt{\frac{a(1 - \cos\theta)}{a(1 + \cos\theta)}}\, dy$$

This parametric equation is then created as we want to simplify the integral to something that cancels out the square root terms and makes it much easier to integrate. We can then substitute this to the integral to get the following expression above.

$$dy = a\sin\theta\, d\theta \qquad x = a\int \sqrt{\frac{(\sin^2\theta)(1-\cos\theta)}{1+\cos\theta}}\, d\theta$$

We now differentiate to get our differential expression for $dy$. This then can be substituted in to make our expression now with respect to theta instead of $y$. We will also take out our constant $a$ to simplify further which we can multiply our solution by later.

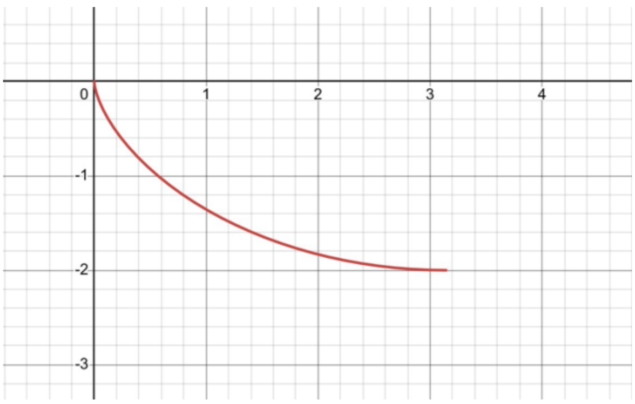
$$\sin^2\theta \ + \ \cos^2\theta = 1 \qquad x = a\int \sqrt{\frac{(1-\cos\theta)^2(1-\cos\theta)}{1+\cos\theta}}\, d\theta$$

Using the trig identity above, we can substitute into the integral making it all in terms of $cos$ before simplifying further to cancel out terms by converting the squared term into two parts

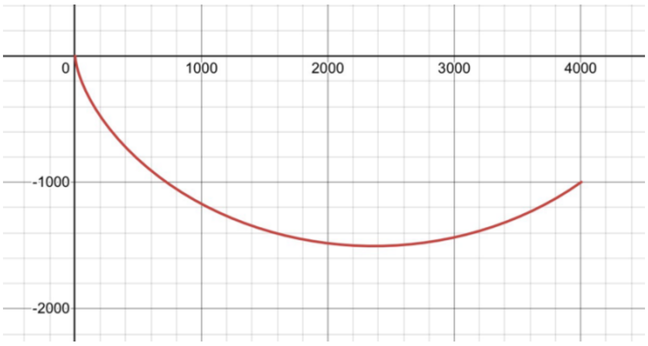
$$x = a\int (1-\cos\theta)\, d\theta$$

$$x = a(\theta - \sin\theta) \qquad y = a(1-\cos\theta)$$

This set of parametric equations describes a cycloid, also known as the brachistochrone. Plotting these equations into a graphing calculator $-\pi < \ t < 0$ where $t$ is theta and $a = \ -1$ produces this graph.




However, there is one part missing still which is where our two points are in our scenario. Since we are looking within the context of skiing, we are going to working in metres. The start to end of the slope within ski mountaineering for example is usually around 4-10 km long and so we are going to take the end point as about 1000m vertically, 4000 metres horizontally. If we put the values of $a = -752$ and $-4.38 < t < 0$ we get a graph that goes from (0, 0) to (4000, -1000).

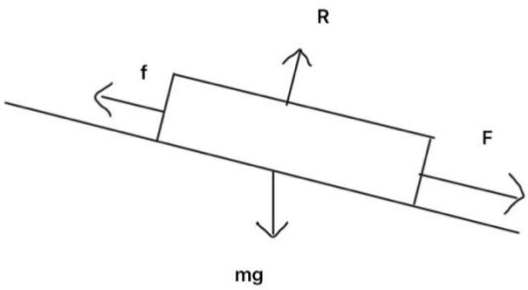


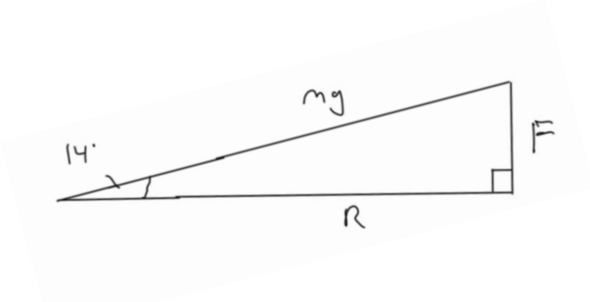

And so, you can see we now have a curve which finds the minimum time between our two points with only gravity acting. This works no matter the mass of the skier due to the conservation of energy however this solution only works only under gravity. So how does this change with the addition of friction?

## INTRODUCING FRICTION

To introduce friction, we are going to consider kinetic friction, which is relatively precise as in an actual scenario, if we considered snow that is cold and dry with rough skis, this would make kinetic friction the main type of friction anyways. We will calculate the various times on a straight line, circular arc, and a cycloid to see how large of a difference friction makes and whether the cycloid is still optimal.

First, we'll find the time for a straight line between the points using the acceleration and distance. Distance is found just using just Pythagoras, giving us $s = 4123.1$ to 2 s.f. For acceleration, we use the formula $F = ma$. To do this we must find the forces on our skier so that we can find the resultant force that can be substituted in.




If we draw a diagram of all forces acting on the skier (who we imagine is a rectangle), we get something like this which can be rearranged so that we are able to calculate the different forces in terms which ones we already know.

$$\sin\theta = \frac{F}{mg} \qquad F = mg\sin\theta$$

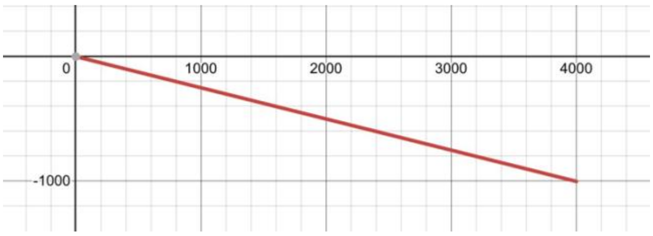
$$\cos\theta = \frac{R}{mg} \qquad R = mg\cos\theta$$

Since the resultant force down the slope is equal to the downwards force minus friction, we need to find what the downwards force and friction are. Using basic trigonometry, we form these two equations which can be used to find the resultant force.

$$f = \mu\, mg\cos\theta \qquad a = \frac{mg\sin\theta - \mu\, mg\cos\theta}{m}$$

$$mg\sin\theta - \mu\, mg\cos\theta \qquad a = g(\sin\theta - \mu\cos\theta)$$

Knowing that $f = \mu R$ we can now minus friction from the downwards force. We have our expression for the resultant force, so we can substitute into $F = ma$ and rearrange to get acceleration.




$$\tan^{-1}(\frac{1}{4}) = 14.04$$

We now need to find the values in the equation staring with the angle of the slope which, if we again consider the two points (0, 0) and (4000, -1000), is about 14 degrees. We also know that $g = 9.81$ as it is just acceleration under gravity.

$$\mu = 0.05 \qquad a = 9.81(\sin(14.04) - 0.05\cos(14.04))$$

Next, we want to find the friction coefficient for skis on snow which would be around 0.05 [4]. Subbing these values in then gives us our value of $a = \ 1.904$

$$s = 4123.1 \qquad u = 0 \qquad a = 1.9$$

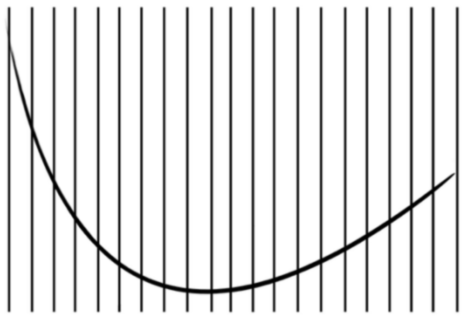
$$s = ut + \frac{1}{2}at^2 \qquad t = 65.8$$

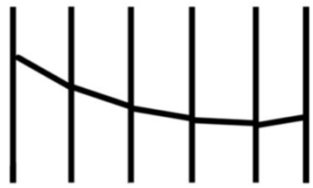
Using the fact $a = 1.9$ to 2 s.f we can use this SUVAT equation to find the time. Since we can't have negative time, we get $t = 65.8$ seconds.

Now we have calculated the time for a constant gradient slope, we can now split up the curves to make them in terms, starting with the arc first before the cycloid. To be reasonably accurate, we will break it up both into 20 segments which can each be calculated.

$$(x-2125)^2+y^2=2125^2 \quad y=-\sqrt{2125^2-(x-2125)^2)}$$

To break the circular arc up, we will consider 20 sections of the horizontal distance and then finding the relevant $y$ coordinate. If we start with the equation above for a possible circle that goes through the points and rearrange for, $y$ we get this. Since it is below the $x$ axis, we will take the negative square root.





If we split up the horizontal distance, we get 200m intervals which can be substituted into the equation for $y$ to give what our angle of each segment is. The process of finding the time will be the same as what we did before apart from including the end velocity, which will be the starting velocity of the next segment. Showing this for each segment would take ages and so to make it easier I will make a table below showing the important values of each segment and the final time of it

| SEGMENT | x1 | y1 | x2 | y2 | y length | distance | angle | acceleration | start velocity | end velocity | time | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 200 | -900 | 900 | 922 | 77.5 | 9.47 | - | 132.1 | 13.95 | 40.49 |
| 2 | 200 | -900 | 400 | -1241 | 341 | 395 | 59.6 | 8.21 | 132.1 | 154.7 | 2.76 | |
| 3 | 400 | -1241 | 600 | -1480 | 239 | 312 | 50.1 | 7.21 | 154.8 | 168.7 | 1.93 | |
| 4 | 600 | -1480 | 800 | -1661 | 181 | 270 | 42.1 | 6.21 | 168.7 | 178.4 | 1.55 | |
| 5 | 800 | -1661 | 1000 | -1803 | 142 | 245 | 35.4 | 5.28 | 178.3 | 185.4 | 1.35 | |
| 6 | 1000 | -1803 | 1200 | -1913 | 110 | 228 | 28.8 | 4.3 | 185.4 | 190.6 | 1.21 | |
| 7 | 1200 | -1913 | 1400 | -1997 | 84 | 217 | 22.8 | 3.35 | 190.7 | 194.5 | 1.13 | |
| 8 | 1400 | -1997 | 1600 | -2059 | 62 | 209 | 17.2 | 2.43 | 194.4 | 197.0 | 1.07 | |
| 9 | 1600 | -2059 | 1800 | -2100 | 41 | 204 | 11.6 | 1.54 | 197.0 | 198.6 | 1.03 | |
| 10 | 1800 | -2100 | 2000 | -2121 | 21 | 201 | 6.0 | 0.54 | 198.6 | 199.1 | 1.01 | |
| 11 | 2000 | -2121 | 2200 | -2124 | 3 | 200 | 0.9 | -0.34 | 199.2 | 198.9 | 1.00 | |
| 12 | 2200 | -2124 | 2400 | -2107 | 17 | 201 | 4.9 | -0.35 | 198.8 | 198.4 | 1.01 | |
| 13 | 2400 | -2107 | 2600 | -2071 | 36 | 203 | 10.2 | -1.25 | 198.4 | 197.1 | 1.03 | |
| 14 | 2600 | -2071 | 2800 | -2014 | 57 | 208 | 15.9 | -2.22 | 197.2 | 194.8 | 1.06 | |
| 15 | 2800 | -2014 | 3000 | -1936 | 78 | 215 | 21.3 | -3.11 | 194.8 | 191.3 | 1.11 | |
| 16 | 3000 | -1936 | 3200 | -1833 | 103 | 225 | 27.2 | -4.05 | 191.3 | 186.5 | 1.19 | |
| 17 | 3200 | -1833 | 3400 | -1700 | 133 | 240 | 33.6 | -5.02 | 186.5 | 179.9 | 1.31 | |
| 18 | 3400 | -1700 | 3600 | -1530 | 170 | 262 | 40.4 | -5.98 | 179.9 | 171.0 | 1.50 | |
| 19 | 3600 | -1530 | 3800 | -1308 | 222 | 299 | 48.0 | -6.96 | 171.0 | 158.4 | 1.81 | |
| 20 | 3800 | -1308 | 4000 | -1000 | 308 | 367 | 57.0 | -7.96 | 158.4 | 138.7 | 2.47 | |

If we again add times together, we find the total time for the arc is 40.49 seconds. We can now repeat but for the brachistochrone however finding the corresponding $y$ will be more difficult as we must find the values of $t$ that will give our desired points from the parametric equations.

Since $a$ was our scale factor it will be the same giving the equation above which $t$ must be found using estimates and tables on a calculator to find values close to what we want for $x$ in our parametric equation before being substituted into the second equation for $y$. I will show an extra column for an estimate of $t$ where for each value of $y$ it can be assumed that this process has been applied.

| SEGMENT | x1 | t | y1 | x2 | y2 | y length | distance | angle | acceleration | start velocity | end velocity | time | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 200 | -480 | 480 | 520 | 67.4 | 8.87 | - | 96.0 | 10.83 | 39.40 |
| 2 | 200 | -1.2 | -480 | 400 | -721 | 241 | 313 | 50.3 | 7.24 | 96.0 | 117.3 | 2.94 | |
| 3 | 400 | -1.53 | -721 | 600 | -908 | 187 | 274 | 43.1 | 6.34 | 117.3 | 131.2 | 2.20 | |
| 4 | 600 | -1.78 | -908 | 800 | -1051 | 143 | 246 | 35.6 | 5.31 | 131.2 | 140.8 | 1.81 | |
| 5 | 800 | -1.98 | -1051 | 1000 | -1170 | 119 | 233 | 30.8 | 4.59 | 140.8 | 148.2 | 1.61 | |
| 6 | 1000 | -2.16 | -1170 | 1200 | -1270 | 100 | 224 | 26.6 | 3.95 | 148.2 | 154.1 | 1.48 | |
| 7 | 1200 | -2.33 | -1270 | 1400 | -1345 | 75 | 214 | 20.6 | 2.99 | 154.1 | 158.2 | 1.37 | |
| 8 | 1400 | -2.48 | -1345 | 1600 | -1404 | 59 | 209 | 16.4 | 2.31 | 158.2 | 161.2 | 1.31 | |
| 9 | 1600 | -2.62 | -1404 | 1800 | -1450 | 46 | 205 | 13.0 | 1.72 | 161.2 | 163.3 | 1.26 | |
| 10 | 1800 | -2.76 | -1450 | 2000 | -1482 | 32 | 203 | 9.1 | 1.07 | 163.3 | 164.7 | 1.23 | |
| 11 | 2000 | -2.9 | -1482 | 2200 | -1500 | 18 | 201 | 5.1 | 0.39 | 164.7 | 165.1 | 1.22 | |
| 12 | 2200 | -3.04 | -1500 | 2400 | -1504 | 4 | 200 | 1.1 | - 0.29 | 165.1 | 164.8 | 1.21 | |
| 13 | 2400 | -3.17 | -1504 | 2600 | -1495 | 9 | 200 | 2.6 | - 0.05 | 164.8 | 164.7 | 1.22 | |
| 14 | 2600 | -3.3 | -1495 | 2800 | -1471 | 24 | 201 | 6.8 | - 0.68 | 164.7 | 163.9 | 1.23 | |
| 15 | 2800 | -3.44 | -1471 | 3000 | -1436 | 35 | 203 | 9.9 | - 1.21 | 163.9 | 162.4 | 1.24 | |
| 16 | 3000 | -3.57 | -1436 | 3200 | -1382 | 54 | 207 | 15.1 | - 2.08 | 162.4 | 159.7 | 1.29 | |
| 17 | 3200 | -3.72 | -1382 | 3400 | -1318 | 64 | 210 | 17.7 | - 2.52 | 159.7 | 156.4 | 1.33 | |
| 18 | 3400 | -3.86 | -1318 | 3600 | -1232 | 86 | 218 | 23.3 | - 3.42 | 156.4 | 151.5 | 1.41 | |
| 19 | 3600 | -4.02 | -1232 | 3800 | -1127 | 105 | 226 | 27.7 | - 4.13 | 151.5 | 145.2 | 1.52 | |
| 20 | 3800 | -4.19 | -1127 | 4000 | -997 | 130 | 239 | 33.0 | - 4.94 | 145.2 | 136.9 | 1.69 | |

Adding times together again, we get 39.4 seconds as our time for the cycloid. So now we have our times for all three ways down we can compare them and see if we can draw conclusions.

## CONCLUSION

To start, the brachistochrone is still fastest out of the three at 39.4 even with kinetic friction applied. The circular arc then follows closely with 40.5 behind and finally the straight line being almost double the time at 65.8 seconds. So, we can conclude that the cycloid is still fastest, although the arc is close behind, and that a straight line is surprisingly slow. However, in terms of skiing, a gap of 0.6 seconds is a big gap so the cycloid would still be much faster to go down than the cycloid. Here we can see a trend in the idea of, with friction, that starting steeply works best to minimise time.

We can now draw a conclusion to skiing. Since skiers can often pick different routes, they can choose how and when they gain speed. If we apply the idea of the cycloid being the fastest still will hold true under kinetic friction, then it means skiers should prioritise early acceleration, rather than just finding the shortest route down, to find faster ways down slopes. So, if you ever find yourself needing to ski down a slope in the smallest time possible, and you somehow end up reading this, I hope this helped you!

## REFERENCES

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