The numerology of cardinality

João Marcelo Farias Queiroz de Oliveira

Oxford essay competition, 2026

1 Introdução

Currently, to the curious ones, it is quite easy to find content explaining the "types of infinity", and I think that is because of a (great) Veritasium video explaining the history of mathematics foundation. These generally explore about the same things regarding infinity: they surprise the audience saying the "rationals and naturals" are the "same size" and that the reals and rationals are of "different sizes", exploring the definition of equality of cardinality as the existence of bijections and its implications on the mentioned sets.

Although this is some great content (i really like some of the videos i saw), i really think there are many interesting things to be explored regarding infinity besides this.

In this essay, I will present how the natural numbers define finite cardinalities, how to construct other infinities, and how we may construct a whole number system with numbers that represent, in some way, infinity.

2 Von-neumann natural numbers

Natural numbers are the most intuitive ones, and are known to basically anyone who received an education. But as we transition to higher math, becomes necessary to rigorously define these. A particularly interesting one is the von-neumann construction.

This construction has a really nice interpretation: each natural number represents one finite size. To do this, we start with 0, and we say $0 = \emptyset$, that is, the set with 0 elements. Naturally, we needed one set to do this; so we define $1 = \{0\}$, as the set representing "one element"; then we defined two objects, 0 and 1, so we take $2 = \{0, 1\}$, and then we take $3 = \{0, 1, 2\}$, and $4 = \{0, 1, 2, 3\}$ and so on.

With this in mind, we can define what it means means to a set have some finite size. We say that a set A has cardinality (or size, intuitively) n if, and only if, we can find to each element of A an unique correspondent in n, in such a way that none of the latter neither are left out neither have two elements of A associated with the same $b \in B$. For example, the cardinality of $\{\mathrm{a}, \mathrm{b}\}$ is two because we can set 0 as a "correspondent" of 0 and b as the one of 1. With this, all elements of $\{\mathrm{a}, \mathrm{b}\}$ have an unique "representant" in 2, with no element of the latter left out; then, $\{\mathrm{a}, \mathrm{b}\}$ has cardinality 2.

I think this is a quite pretty and natural definition, for each number not only represents, but also embodies a cardinality; for example, we may say that 2 not only represents the cardinality 2: it is the cardinality 2.

Saying "one-to-one correspondence" becomes quite hurdlesome really quickly, so it will be presented the more rigorous and usable definition of it. So we will use functions to make this less troublesome.

A function $f : A \to B$ is an object where each element of A is associated with one, and only one, element in b; saying this in another way, each element in A is associated with one in B, in such a way there are not two distinct $x, y \in B$ such that x, y are associated with the same $a \in A$; we say that the element $b \in B$ associated with $a \in A$ is $f(a)$. Notice that this definition does not imply each element of B is associated with only one element of A.

Coming back to cardinalities, I spoke a lot about "correspondence" between sets, and said that two sets have the same cardinality if they obey some conditions; these conditions are the same of saying that there is function $f : A \to B$ that we call bijective. A bijective function (or bijection) is a function where: each element of B has a unique element of the domain that is "sent" to it; and every element of B has a element $a \in A$ where $f(a) = b$.

3 Infinite cardinalities

Besides finite sizes, there are infinity sizes too; these are defined to be the cardinalities that are not finite.

We know no inifinity set is smaller then the natural numbers. That is because, for any set A, we may take a function $f : A \to \mathbb{N}$ where $f(a_0) = 0$ for some $a_0 \in A$, and $f(a_1) = 1$ for $a_1 \neq a_0$, and so on. Then, this either ends at $f(a_n) = n$, case where the cardinality of A is $n + 1$, or it never ends and f is a bijection to the natural numbers.

From this, a question we might ask is: what is the next greatest cardinality? And the answer is that we cannot know, for this is unprovable in standard set theory. But we commonly assume there is one, that will be shown here.

Before seeing it, I should introduce a seemingly obvious concept: exponentiation, an idea that relies on multiplication.

3.1 Multiplication

Now, think about what we do when we multiply $n, m \in \mathbb{N}$; we take n and sum it m times. In sets, it is done the same: the product between two sets A and B is: $A \times B = \{(a, b) \in C | a \in A, b \in B\}$, where (a, b) is a arbritary set that we pick such that $a \neq b, (a, b) \neq (b, a)$ (that is, a set where the order of a and b matters), where C is here only for technical reasons, and there is no need for concerning about it. This reflects multiplication in the sense we take, for each b, all the elements of a and associate them with it, just like we do to natural numbers: when multiplying n and m, we take the first one in the sum $n = 1 + 1 + ... + 1$ and repeat it for each one forming $m = 1 + 1 + ... + 1$, giving us $((1 + ... + 1)(n \text{ times}) + (1 + ... + 1)(n \text{ times}) + ... + (1 + ... + 1)(n \text{ times}))(m \text{ times})$. In this case, multiplicating n and m as we defined them gives us $(0, 0), (0, 1)...(0, m-1), (1, 0)...(n-1, m-1)$, so we take each unity composing m and repeat is n times quite literally indeed. Please note this construction work for A and B of any cardinality, and the natural numbers were just an intuitive example.

3.2 Exponentiation

With this, let's introduce exponentiation. Now, elevating n to m is only repeating the multiplication of n by itself m times. And that is just what we do! A^B is just $A \times A \times A... \times A$ repeating the operation $\times$ for each element of B, but in slightly different way. The natural numbers will, again, be used for the sake of clarity. When we exponentiate n and m, it is the same as multiplying n by itself m times, and that is the way we define exponentiation of sets, with a caviat. In this case, as n and m were defined to be set, we need to construct the set with its elements being $(n_0, ..., n_{m-1})$, that is, a construct where there is m places and the order matters. But this is the same to say that, for each element of m, with m elements, there is an element of

n associated with it, and that for $x, y \in n$ and $z, w \in B$, it matters whether we associate x with z and y with w or x with w and y with z.

Notice that we just described functions with domain B and range A; a function $f : B \to n$ is such that it matters whether $f(z) = x$ or $f(z) = y$, and each element of B works as a "place" where we put an element of A. Thus, A^B is the set of functions $f : B \to A$, expanding again a concept from natural numbers to sets.

3.3 What about cardinality?

Somethings were defined, but a question that could arrise is what it has to do with cardinality. To start, it will be defined inequalities between cardinalities. I will start to denote the cardinality of a set A as $\#A$.

We say $\#A \leq \#B$, if, and only if, there is a bijection from A to a subset of B, and this inequality is strict, that is, $\#A < \#B$, if, and only if, $\#A \leq \#B$ and there is not a bijection from A to B. As an example, $\#3 \leq \#3$ because there is a bijection from 3 to the subset 3 of 3, and $\#2 < \#3$ because there is a bijection from 2 to the subset 2 of 3, but not to the whole set 3.

Then, we can notice something: the set of all the subsets of a set A, denoted by $\mathbf{P}(A)$, is such that $\#P(A) > \#A$ (this is a set assumed to exist in set theory, not being necessary to prove its existence). This can be a somewhat hard to understand proof, and the reader who wants to jump it can do so without affecting the general understanding.

To see that, assume otherwise and let $f : A \to \mathbf{P}(A)$ be a bijection from A to $\mathbf{P(A)}$. Then, consider the set $X = \{a \in A | a \notin f(a)\}$. As every element of $\mathbf{P}(A)$ would have a correspondent in A, there would be x such that $f(x) = X$. If $x \in X$, then of course, because $x \in X$, $x \notin f(x) = X$, a contradiction. But if $x \notin X$, then $x \notin f(x)$ which implies that $x \in X = f(x)$, also a contradiction.

Besides that, there actually is a bijection between 2^A and $\mathbf{P}(A)$: consider the function $f : 2^A \to \mathbf{P}(A)$ where $f(g) = \{a \in A | g(a) = 1\}$. Then we verify that, for each element of 2^A there is only one $f(g)$, and that, if $f(g) = f(h)$, then g and h must be equal.

This is a result that connects the concept of exponentiation and infinite cardinality. It actually follows that $\#\mathbb{R} = \#2^{\mathbb{N}}$, which is, unfortunately, out of the scope of this essay.

So, we can construct as many infinities as we want taking $\mathbf{P}(\mathbb{N})$, and then $\mathbf{P}(\mathbf{P}(\mathbb{N}))$ and so on.

4 The ordinal numbers

Besides the usual numbers normally talked about, there are the ordinal numbers. Those may be thought of as an extension of the von-neumann natural numbers, where they possess numbers representing infinity.

Firstly, notice that the natural numbers follow a rule: they start with the empty set, and define that, for each number $n \in \mathbb{N}$, the successor $s(n) = n \cup n$. This is actually an axiom, where this is a set assumed to exist.

We now will construct the first ordinal number as an example: $\omega = \mathbb{N}$. Then, we set $\omega + 1 = \omega \cup \{\omega\}$, and $\omega + 2 = (\omega + 1) \cup \{\omega + 1\}$ and so on. Unfortunately, the more rigorous definition of ordinal numbers cannot be shown here; but they may be thought of as any set that comes from this opertion $\alpha \cup \{\alpha\}$, with α being a set of ordinals or an ordinal number; also we say that all natural numbers are also an ordinal number, that the ordinal numbers are totally ordered (every two distinct ordinals are such that either $\alpha < \beta$ or $\beta < \alpha$) by $\in$; that is, $\alpha < \beta$ if and only if $\alpha \in \beta$, and that all subsets of ordinal numbers are well-ordered: all of its subsets have a "least element".

Now, it will be given a intuition of what the ordinal numbers are. Each one of them represent a element that is in a "position" in a ordered set. For example, taking the set {1,2,3}, the number 3 is ordered at the third position, or the position that 2 is in the ordinal numbers. Similarly, omega represents a number that is after all the natural numbers.