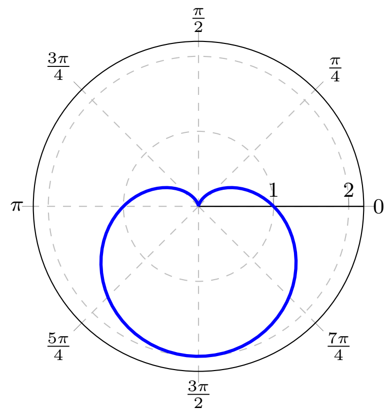
Making spirals

Thomas Etchegaray

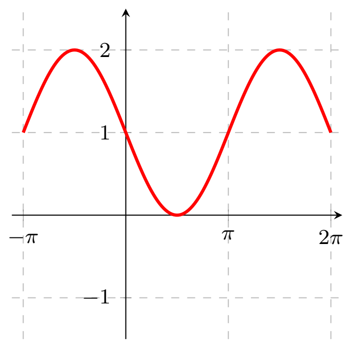
Paris, April 2026

Most of our life, we have graphed all kinds of lines and spirals on grids, specifically, the cartesian grid denoted $\mathbb{R}^2$. The furthest I went way from this comfort was when I learnt about polar coordinates in my calculus class, I was amazed to find this new way of graphing and found it really satisfying how we attained every point of the plane using a line that grew smaller or bigger and that turned to a certain angle, much more satisfying and unique than simply having x-coordinates and y-coordinates!

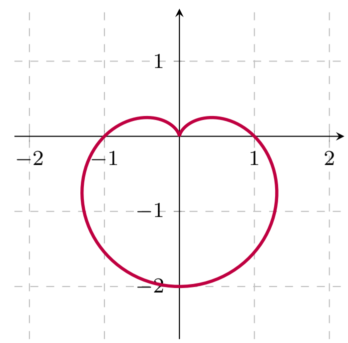
Here are some nice graphs on both polar coordinates and cartesian coordinates:



Polar: $r = 1 - \sin(\theta)$



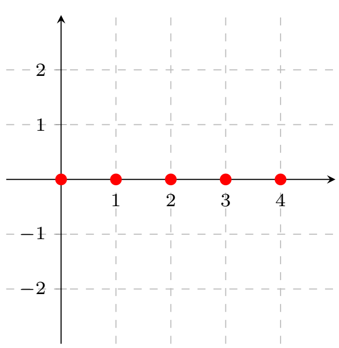
Cartesian: $y = 1 - \sin(x)$



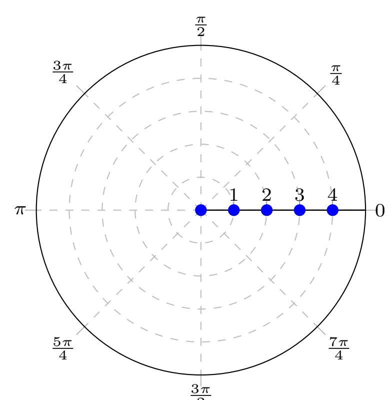
Cartesian:
$(x^2 + y^2 + y)^2 = x^2 + y^2$

As you can see the same equation will turn into a beautiful heart on polar coordinates, a sine curve on the cartesian plane and to get that same heart on the cartesian plane, we need a pretty hefty implicit equation. This makes us think about what else could be extremely different from cartesian coordinates to polar coordinates. But instead of plotting equations like we did before, let's instead plot points manually.

Let's start simply, maybe just plotting the points $S = \{(0,0);(1,0);(2,0);\ldots(4,0)\}$. This means that we will be plotting points who have length $r = n$ and angle from origin $\theta = 0$.

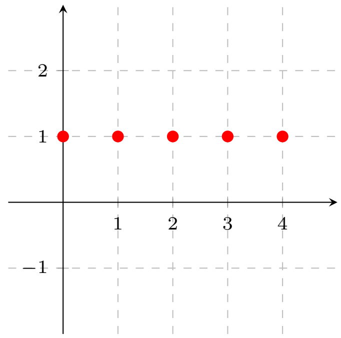


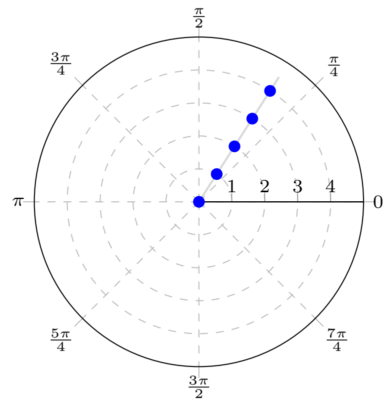
Cartesian Plane
$S = \{(x,y) \mid y = 0, x \in \{0,1,2,3,4\}\}$



Polar Plane
$S = \{(r,\theta) \mid \theta = 0, r \in \{0,1,2,3,4\}\}$

Here, the graphs look exactly the same which is not our goal, let's try plotting $S = \{(0,1);(1,1);(2,1);\ldots(4,1)\}$.



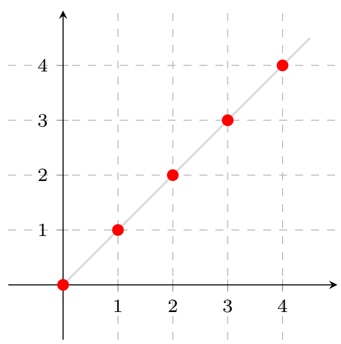


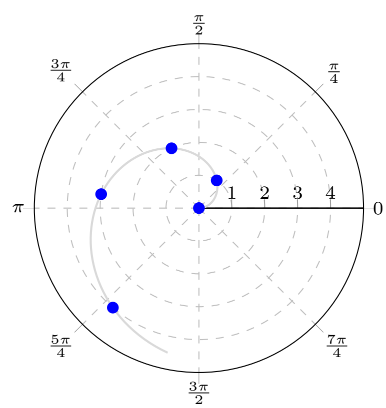
Cartesian Plane
$S = \{(x, y) \mid y = 1, x \in \{0, 1, 2, 3, 4\}\}$

Polar Plane
$S = \{(r, \theta) \mid \theta = 1, r \in \{0, 1, 2, 3, 4\}\}$

I notice that simply adding one to all of the y-coordinates and angles θ starts creating differences in the plots, the points on the cartesian plane are simply shifted up by one while the points on the polar plane are rotated by a certain angle $\theta = 1\,\text{rad}$.

This makes me think, what if I change the coordinates once again but not by one each time but by n with $n \in [\![1, n]\!]$, if the transformations I described hold, the points on the cartesian plane should rotate (by $\frac{\pi}{2}$) and the points on the polar plane should rotate even more but I'm not sure how, let's see!

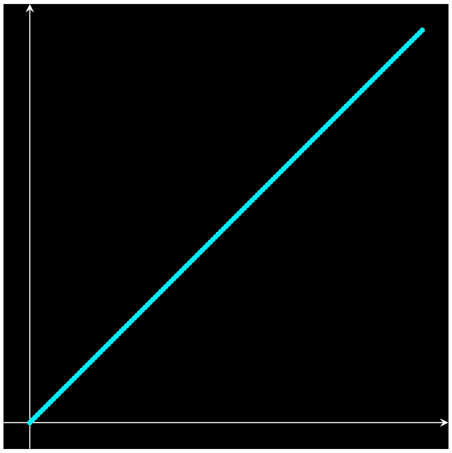




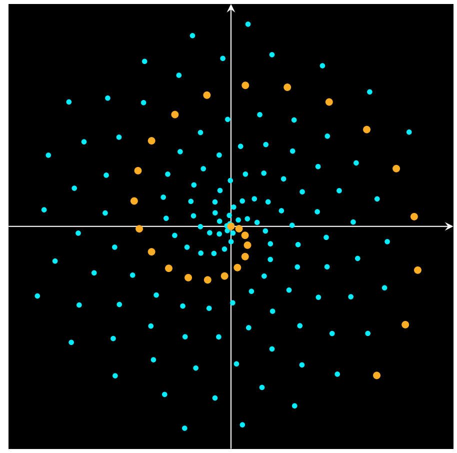
Cartesian Plane
$S = \{(x, y) \mid x = n, y = n, n \in [\![0, 4]\!]\}$

Polar Plane
$S = \{(r, \theta) \mid r = n, \theta = n, n \in [\![0, 4]\!]\}$

Exactly! the plots on the cartesian plane rotated to make an increasing line while the points on the polar plane transformed into some sort of spiral that looks really nice. Let's plot more points and see how it looks like, the straight line will only get more boring but the spiral could turn into something really cool!

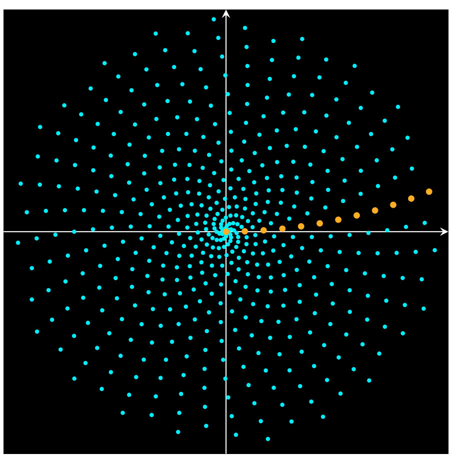


Cartesian Plane
$S = \{(x, y) \mid x = n, y = n, n \in [\![0, 150]\!]\}$

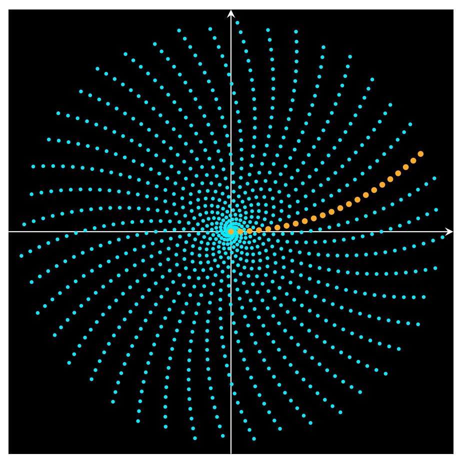


Polar Plane
$S = \{(r, \theta) \mid r = n, \theta = n, n \in [\![0, 150]\!]\}$

On the polar plane, the points form a magnificent spiral, and as expected, the diagonal is the same on the cartesian plane. So for now, let's only focus on the polar plane as it has much more to offer. The first thing that comes to my mind is how are those spirals created ? Another interesting thing is that there seems to be 6 different spirals (I highlighted one of the spirals in orange), will there always be 6 spirals? A good way to get more intuition might be to zoom out even more, let's try 500 and 1000 points.

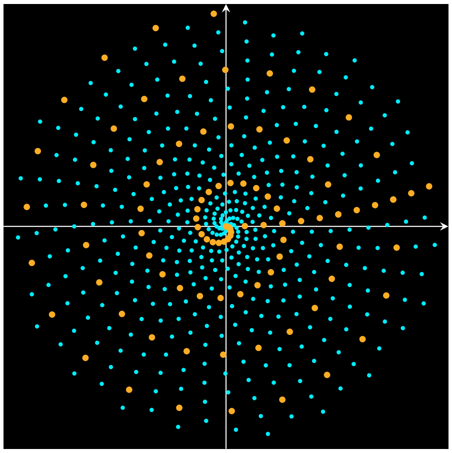


Polar Plane
$S = \{(r, \theta) \mid r = n, \theta = n, n \in [\![0, 500]\!]\}$

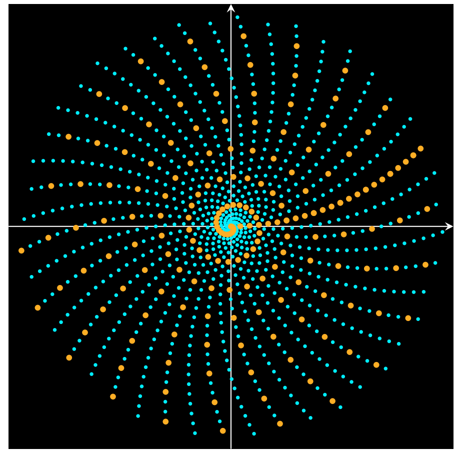


Polar Plane
$S = \{(r, \theta) \mid r = n, \theta = n, n \in [\![0, 1000]\!]\}$

Nice! There are still some beautiful spirals but it seems like they have changed, counting them, there seems to be 44 spirals. If we look closely at the center, there still seems to be the 6 spirals that we saw on the previous graph but those 6 spirals seem harder to follow as soon as we add more points. Another interesting thing is that even though there is a huge difference between plotting 500 points and 1000 points, there are still exactly 44 spirals, will this number change to a bigger number like it did before from 6 to 44 ? However, if we look closely, are there actually 44 spirals or are there 6 of them that are so curled up that we can't even distinguish them anymore ? I'll also highlight one of the original 6-spiral and see if that makes it clearer.



Polar Plane
$S = \{(r, \theta) \mid r = n, \theta = n, n \in [\![0, 500]\!]\}$



Polar Plane
$S = \{(r, \theta) \mid r = n, \theta = n, n \in [\![0, 1000]\!]\}$

Wonderful! As shown by the newly highlighted orange points, we can still distinguish one of the original 6-spiral, but unless highlighted it is much harder to distinguish it from the 44-spirals. Why is that ? Well if you look closely, as the points have a distance r and angle θ that is close to 500, then the distance between 2 points that are supposed to be next to each other on a 6-spiral is actually much larger than the distance between 2 points that are next to each other on a 44-spiral. And because our eyes form groups (in this case spirals) from the object or shapes that are closest to each other (this is also known as the Gestalt law of proximity) our brain is tricked into thinking there are 44 spirals.

Alright so now we understand why we see 44 spirals instead of 6, however we still don't know how those spirals are created and we still don't know what's so special about the numbers 6 and 44. Once again we'll plot more points to get a better understanding, let's go much bigger this time, around 35 000 points:

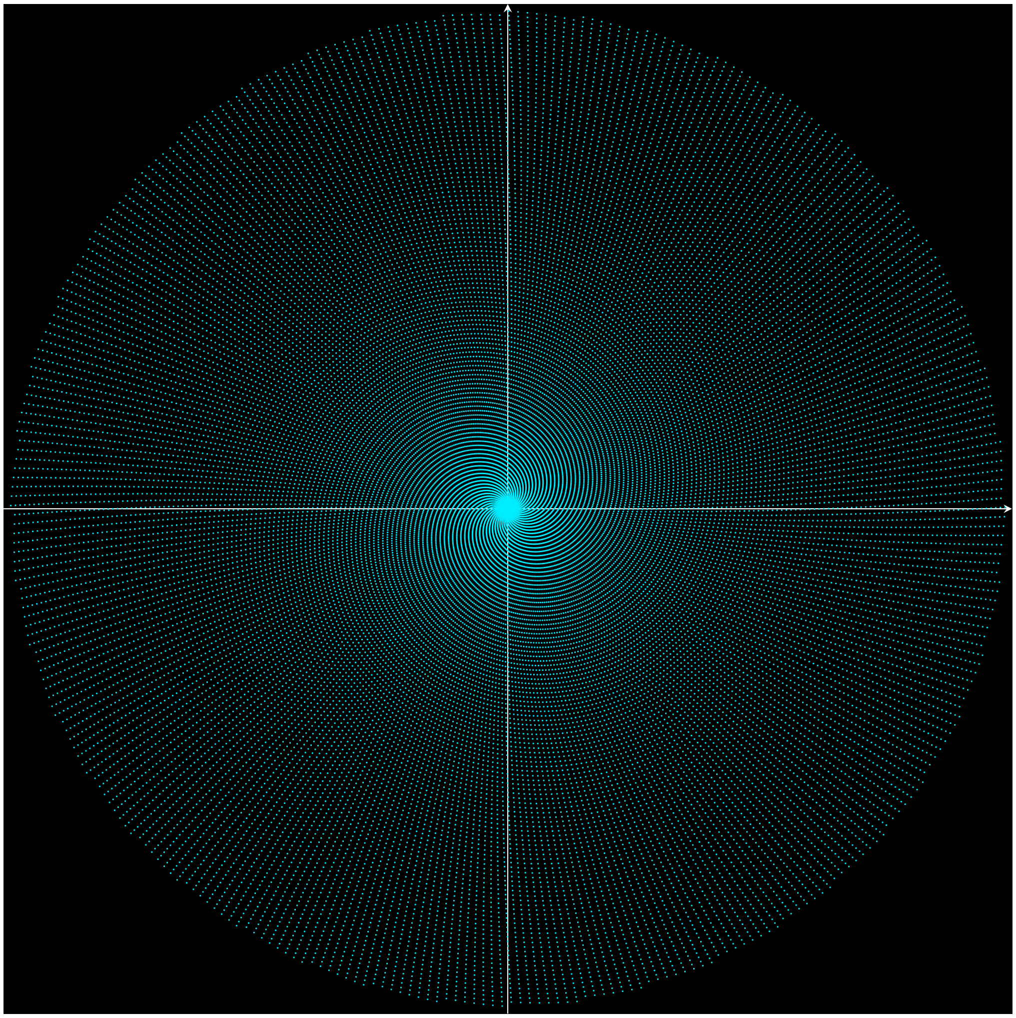


Figure 1: Polar Plane for $n \in [\![0, 35500]\!]$

Wow, that is truly mind blowing. Now we only see spirals towards the center and then we see what seems to be perfectly straight lines, 333 lines if you count them. Now this is fun. We went from 6 lines to 44 lines to 333 lines! What on earth is the link between those numbers and how are these spirals (or should I say lines) created ?

Well it turns out the answer is common. You might have guessed it as we have working with angles all along, but it's all because of π! or his big brother $\tau = 2\pi$. Basically, every time we increment n by 1, we rotate our point by exactly 1 radian. And the reason we got those specific numbers spirals if because $2\pi \approx 6.2832$, so after 6 steps we have rotated by 6 radians, which is *almost* a full turn. We fall short by $2\pi - 6 \approx 0.2832$ radians. This means that the point $n = 6$ lands slightly clockwise of the point $n = 0$ (viewed from further out, and with a larger radius). The point $n = 12$ lands slightly clockwise of $n = 6$, and so on. Connect them up and you get one of the 6 arms, all curling inward because each step of 6 falls short of a full turn.

That is the whole secret. The 6 arms exist because $6 \approx 2\pi$. But why 44? Why 333? Why 710?

We started with 6 because it intuitively is the closest integer to 2π but it turns out it is also there because it is the first value for the continued fraction of 2π. Continued fractions are wonderful and I really enjoyed learning about them will writing this essay but unfortunately I won't have time to go on about them but definitely have a look at it. Here is the start of the continued fraction for 2π:

$$2\pi = 6 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{7 + \cfrac{1}{2 + \cfrac{1}{146 + \cfrac{1}{3 + \cdots}}}}}}}$$

And we got our 44 and 333 spirals by cutting this fraction :

$$6 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{1}}} = \frac{44}{7} \qquad 6 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{7}}}} = \frac{333}{53} \qquad 6 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{7 + \cfrac{1}{2}}}}} = \frac{710}{113}$$

We call these numbers convergents and each convergent p/q is the best rational approximation to 2π with denominator at most q, and each tells us that q steps of 1 radian is almost exactly p full turns. So after q points, we land almost where we started. This creates p visible spiral arms.

Convergent p/q	Arms visible	Step along arm	Angular error per q steps
6/1	6	1	≈ 0.283 rad
44/7	44	7	≈ -0.0088 rad
333/53	333	53	≈ 0.0094 rad

We can instantly notice that each convergent is a better approximation than the last as the angular error shrinks rapidly. This is why the arms straighten out as we zoom out: at large n, the tiny error of the 44/7 approximation becomes visible as spiraling, but the even tinier error of 333/53 does not, so 333 arms look perfectly straight.

Now, everything we have done depended on 2π having great rational approximations, each better than the last. But what would happen if we chose a step angle whose rational approximations were as bad as possible?

Let's use the golden ratio $\varphi = (1 + \sqrt{5})/2$ has continued fraction expansion

$$\varphi = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}},$$

with every coefficient equal to 1. This is as small as continued fraction coefficients can get, and it makes φ the hardest real number to approximate by rationals because each convergent is only slightly better than the one before it, which is the opposite of what we saw with 2π.

Let's redo our plot, but instead of stepping by 1 radian each time, we step by $2\pi/\varphi \approx 3.883$ radians, the golden angle:

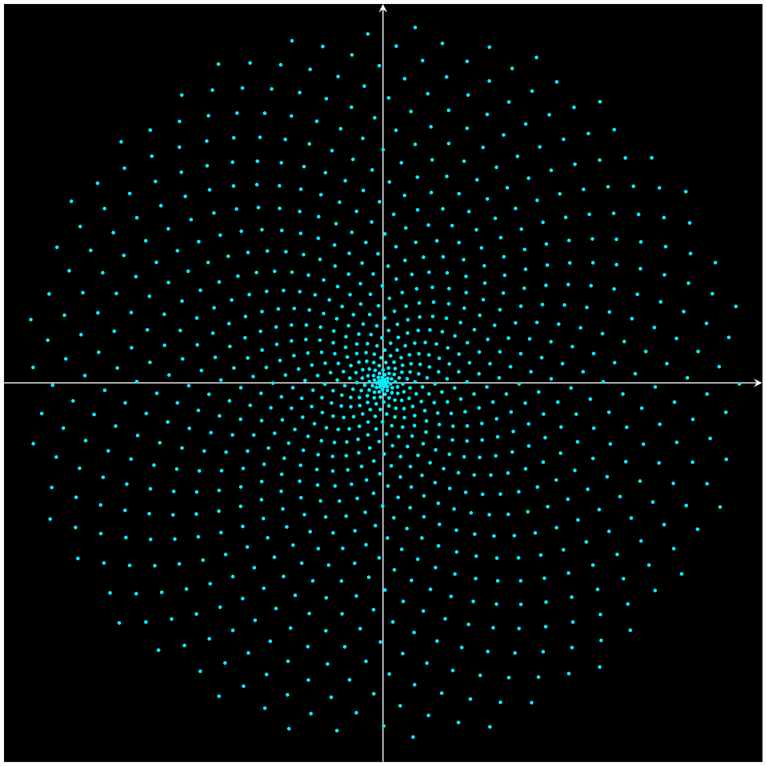


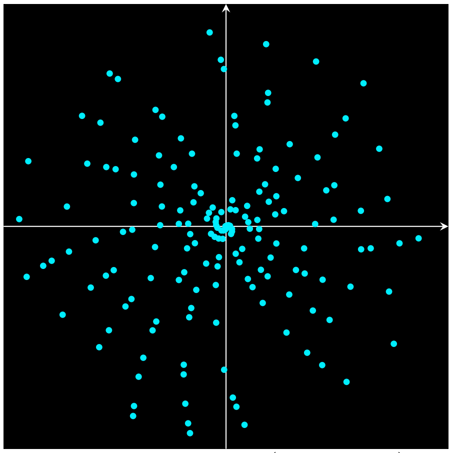
Figure 2: Polar plane with step angle $2\pi/\varphi$, for $n \in [\![0, 1000]\!]$

Compare this to the 1-radian plot at the same point count, which showed a perfect 44-spiral pattern. Here, with the golden angle, there is no specific arm count that we can observe, the points are just distributed around the spiral as evenly as they can.

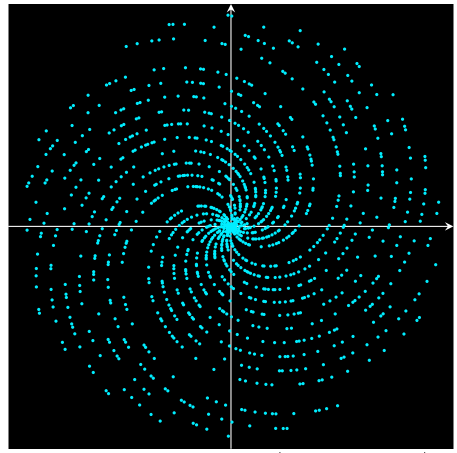
This is exactly what we expected because of φ's continued fraction! With 2π, each convergent produced a window where a specific number of arms dominated before the next convergent took over. With the golden angle, the convergents are so close that no single one ever gets to dominate which means the pattern never settles on a specific number of arms.

This is also the reason we find the golden angle in plants. For example, sunflower seeds are arranged at angles of $2\pi/\varphi$ from one another, which is precisely the arrangement that avoids creating sections where sunlight or space would be wasted. Any other angle would create spirals or lines or gaps at some scale, only the golden angle constantly avoids this.

So far we have plotted every integer from 0 to n. Now, let's let our intrusive thoughts take control, what if we only plot the n that are prime ($r = p$, $\theta = p$ in radians with $p \in [\![0, p]\!]$).



Primes up to 1000 (168 points)



Primes up to 10,000 (1,229 points)

Surprisingly, the spirals are still there! In both plots we can count 20 spirals. In the second plot they start curling a lot, this is the same effect we saw before. But what actually changed is that there are only 20 arms, not 44. Many of the arms are missing entirely.

Why? Let's think about which points a spiral has. We saw that the 44-spiral pattern comes from stepping n by 44 gives you to almost the same ray. So the spiral through $n = k$ contains the points $n = k, k+44, k+88, k+132, \ldots$ which can also be written as all n with $n \equiv k \pmod{44}$.

But now if $\gcd(k, 44) > 1$, then every number in that spiral shares a common factor with 44, and so cannot be prime (except possibly k itself, if k is one of the prime factors of 44). Since $44 = 2^2 \cdot 11$, any $n \equiv k \pmod{44}$ with k even or divisible by 11 is itself even or divisible by 11, and thus composite.

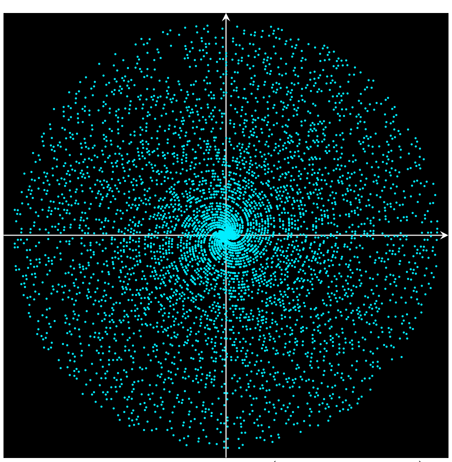
So out of the 44 possible remainders $k \in \{0, 1, 2, \ldots, 43\}$, the only ones that can contain infinitely many primes are those coprime to 44. The count of these is given by Euler's wonderful totient function:

$$\varphi(44) = 44 \cdot \left(1 - \tfrac{1}{2}\right)\left(1 - \tfrac{1}{11}\right) = 20.$$

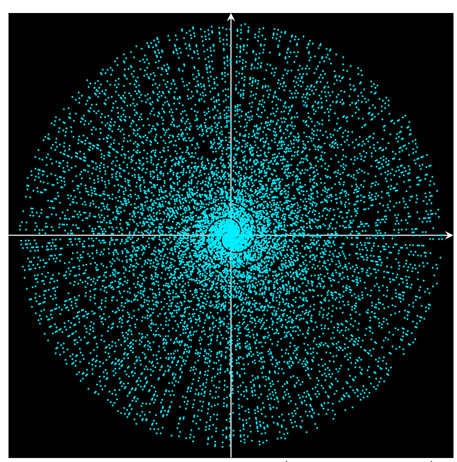
Exactly 20 spirals out of 44 survive which is exactly what we see!

Of course, the same reasoning works at other ranges of points, at the 6-spiral scale: $\varphi(6) = 2$. Only the arms $n \equiv 1$ and $n \equiv 5 \pmod 6$ contain primes. At the 333-arm scale: $333 = 3^2 \cdot 37$, so $\varphi(333) = 216$ arms survive.

Like we did before, to make sure our math is right, let's plot many more primes:



Primes up to 50,000 (5,133 points)



Primes up to 100,000 (9,592 points)

Beautiful! once again what we predicted. At these scales the inner core still shows the 20 spirals from before, but further out if we take time to count them, there are the 216 surviving spirals of the 333-spiral pattern.

We have just shown which arms can contain primes. But this can make us think of another thing: among those surviving spirals, is the distribution of primes uniform? Are there as many primes $\equiv 1 \pmod{44}$ as there are $\equiv 3 \pmod{44}$, in the long run?

Looking at the plots, the surviving arms do seem roughly equally bright. After looking into it, I found it is a theorem that was proved by Dirichlet: For any integers a and q with $\gcd(a, q) = 1$, there are infinitely many primes p with $p \equiv a \pmod{q}$. And in the long run, the primes are distributed equally out of the $\varphi(q)$ possible remainders.

Now I will end with one math curiosity, Dirichlet's theorem tells us that primes are distributed evenly across the allowed spirals as n goes to infinity. But for any finite range, there should be small differences right ? For example, primes $\equiv 3 \pmod 4$ slightly outnumber primes $\equiv 1 \pmod 4$ for almost every value n we choose as long as n isn't astronomically large. This is called Chebyshev's bias, and it is still unfortunately not fully understood. I really hope someone will soon make sense of it so I can end this essay the right way!

Note: the video "Why do prime numbers make these spirals? | Dirichlet's theorem and pi approximations" created by 3Blue1Brown was really useful for writing this essay, it is what gave me the idea to write about this beautiful subject even though I tried to tell the story differently, in my own way and taking time to mention other things I found really interesting.

Thank you for reading!