Paradoxical Coastlines – Fractals in Nature

Thomas James Melia

April 2026

1 Introduction – The Coastline Paradox

Throughout nature, we may find certain phenomena which seem to have no sensible or logical explanation. Allow us to take an example of this – the coastline paradox. It may seem to be an obvious result of taking measures of the lengths of coastlines that, no matter what, the total length should approach a finite value – some sort of limit. However, as I intend to prove, this length is **infinite**.

First, let us draw our attention to the mathematical concepts of *fractals*. Coined by the pioneer of the subject, Benoit B. Mandelbrot, the word "fractal" originates from the Latin word "fractus," referring to something that is fractured or broken apart. Fractal geometry allows us to find approximates for different measures of shapes that may be impossible to find otherwise, due to a lack of perfect precision with measurements. Because of this, we may rather use fractal geometry to prove our initial statement.

2 Foundations of Fractality

What governs fractality? For our coastline problem, we need only identify three main features as per Mandelbrot's definition. To bring out these properties, let there exist a circle with its centre at the origin of a standard x-y plane. We define our circle's equation as $x^2 + y^2 = r^2$, where our starting radius is 1. We will proceed to halve, and double, the radius an infinite number of times. This process is an example of iteration – one of our first requirements for fractality – and to generalise this, we introduce a variable n which represents the place of such circle in the series. This gives us an equation of $x^2 + y^2 = 2^{-n}$. We see an example of this through **figure 1** on the following page. What do we notice?

2.1 Properties of Fractals

No matter how much we zoom in or out, we will always return to what seems to be our starting position, merely at different magnifications. We shall refer to this property as *self-similarity*, which denotes repetition across several scales. This is quite an obvious feature of our circles and is another of Mandelbrot's requirements. Lastly, we see that, upon closer inspection, we never reach a point at which new circles no longer appear at the centre. This is an example of infinite detail, which is our last governing feature for fractality.

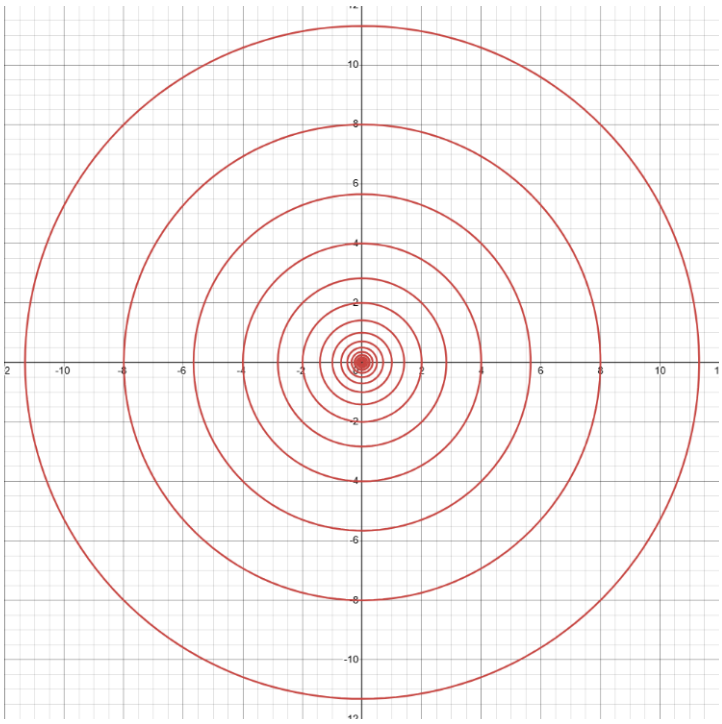

Figure 1: Series of concentric circles with equations $x^2 + y^2 = 2^{-n}$.

2.2 Additional Examples

To elaborate briefly, we turn to the *middle third Cantor Set*. This set of real numbers is taken from the interval $[0,1]$, where $0 \leq x \leq 1$. We divide the interval into 3 smaller, equal intervals and remove the middle section such that we are left with the intervals $\left[0, \frac{1}{3}\right]$ and $\left[\frac{2}{3}, 1\right]$. We repeat this infinitely to form our Cantor Set through iteration. As you may notice, this links back to our example of the coastline as examining the set closely reveals finer details of smaller, self-similar intervals, giving us the notion of fractality.

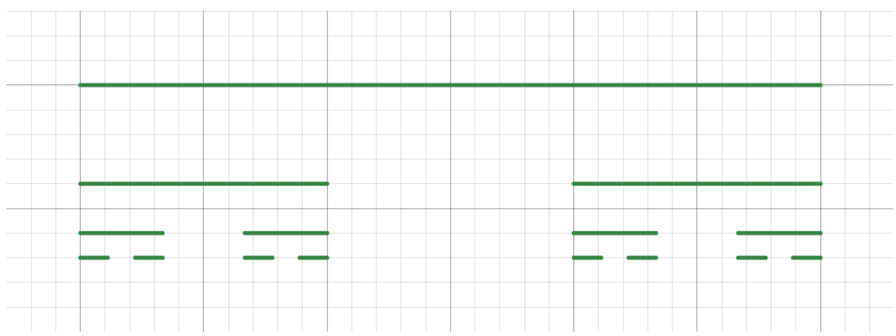

Figure 2: Middle third Cantor Set (up to step 4).

3 Methods for Finding Properties of Fractals

To get closer to understanding this coastline paradox, we need to investigate past standard Euclidean geometry of integer dimensions – but how could a dimension be of a non-integer value, say 1.5? Let us first understand Lebesgue measure.

Lebesgue measure gives a sense of length to collections of subsets, where subsets refer to groups of values taken from any larger group. To simplify the concept of a 'set' or 'subset,' take a 10 unit-long rope of negligible width and lay it flat along the x-axis. As every point of the rope lies on the x-axis such that $(0,0)$ and $(10,0)$ are also coordinates, we may write the interval of all possible values as $[0,10]$, where $0 \leq x \leq 10$. This interval is our *set* of values for all coordinates of points on the rope. Now say that we take only a section of the rope, say $[2,6]$. We call this a *subset* as it pertains to a larger set which contains the specified interval. We use the symbol $\subset$ to represent a subset.

If you wanted to determine the length of an interval, you would simply take such an interval, say $[a, b]$, and use $b - a$ to find that length. We refer to this length as the Lebesgue measure of length of a set or collection of subsets such that $\mathcal{L}^1[a, b] = b - a$, where $\mathcal{L}^n$ refers to Lebesgue measure in n dimensions. Hence, $\mathcal{L}^1 =$ length, $\mathcal{L}^2 =$ area, and so on. We also use this to determine lengths that cannot be found through simpler means, such as that of our coastline.

Consider A to be a collection of separated or *disjoint* intervals such that we may describe A as the countable union of disjoint intervals $[a_i, b_i]$, where the i refers to the interval's place in the set. As these intervals are separate and thus have individual lengths, we need to summate all intervals within our union to find a Lebesgue measure for length of A. Our earlier use of the term *countable* merely relates to the idea that all elements in the set can be listed in an order $(a_1, a_2, \dots)$. We will use the notation of $\cup_i$ to represent our countable union such that we may describe A as

$$A = \cup_i [a_i, b_i].$$

Thus, we can summate the Lebesgue measure of length of A as

$$\mathcal{L}^1(A) = \sum (b_i - a_i).$$

3.1 Technical Definition

However, this does not account for overlapping intervals. We use the term 'infimum' to represent the least possible number of sets needed to construct some larger set. Think of the infimum as the lower limit of a length or number; we shall use the notation inf to refer to this. To arrive at a more technical definition of the Lebesgue measure of length of A, we take the infimum of the summation of all lengths in set A such that A is a subset of all countable unions of intervals $[a_i, b_i]$. The following shows the relationship as described.

$$\mathcal{L}^1(A) = \inf\{\sum_{i=1}^{\infty}(b_i - a_i) : A \subset \bigcup_{i=1}^{\infty}[a_i, b_i]\}$$

4 Hausdorff Dimensions

Now that we have a technical definition for the total length of a set, we can begin to understand how non-integer dimensions exist, as we have sought to understand. We may think not only of the box-counting dimension, but also of more fundamental dimensions such as Hausdorff dimension, as how space-occupying a set really is. Hausdorff dimension is more commonly used to find non-integer dimensions of simpler fractals which, for the most part, obey self-similarity. Let us understand a formula for Hausdorff dimension.

Take a square of dimension 2, downscale by ratio 4 such that we are left with 16 identical copies. Let $m =$ number of copies, $D =$ dimension, $r =$ downscale ratio.

Take $m = r^D$ such that m obeys a power law.

$m = 4^2 = 16$, as required.

This shows that there is a clear relationship between the variables, and thus we can rearrange to make D the subject.

Take natural logs (or just logs) of both sides.

$$\ln m = \ln r^D$$

$$\ln m = D \ln r$$

$$D = \frac{\ln m}{\ln r}$$

To further assure us of the accuracy of this equation, let us use our example of the square.

$$D = \frac{\ln 16}{\ln 4} = \frac{2 \ln 4}{\ln 4} = 2, \text{ as required.}$$

For the middle third Cantor Set, we may now use this simplified equation to find a Hausdorff dimension and thus a Lebesgue measure for length of the set. Let us define our variables. Downscale ratio of 3 as we split the interval into 3 smaller, identical intervals, of which we only keep 2. Thus, $m = 2, r = 3$.

$$D = \frac{\ln m}{\ln r}$$

$$D = \frac{\ln 2}{\ln 3}$$

$$\Longrightarrow D = 0.630929753571 \ldots \approx 0.631 \ (3 \ d.p)$$

4.1 Properties of Hausdorff Dimensions

What can we infer from this newly obtained value? If we were to round this value *down* to the nearest integer, we would get 0. Since we now know that Hausdorff dimensions show how densely a fractal occupies a space, we can state that the Lebesgue measure of length of the middle third Cantor Set is 0. This is because, as the dimensions of objects may not exceed their plane of residence, the dimension cannot be greater than 1, and hence, with our new formula, we will see that the total Lebesgue measure of length of remaining intervals is 0.

To prove this, take the Lebesgue measure of length of all *removed* intervals to estimate the remaining length. Let C represent this.

As lengths of removed intervals start from $\frac{1}{3}$ and we remove intervals each time such that the amount is a multiple of 2 starting from 2^0, we have $\frac{2^{n-1}}{3^n}$.

To organise these relationships in an equation, we establish the following:

$$\mathcal{L}^1(C) = \inf\{\sum_{n=1}^{\infty}\left(\frac{2^{n-1}}{3^n}\right) : C \subset \bigcup_{n=1}^{\infty}[0,1]\}$$

$$\mathcal{L}^1(C) = \sum_{n=1}^{\infty}\left(\frac{2^{n-1}}{3^n}\right)$$

$$\mathcal{L}^1(C) = \frac{1}{3}\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^{n-1}$$

$$\mathcal{L}^1 = \frac{1}{3}(3) = 1$$

Hence, the total length is $\mathcal{L}^1[0,1] - 1 = 1 - 1 = 0$, as stated.

5 Box-Counting Dimensions

Now that we know how to use Lebesgue measure of length, let us move onto what brings our investigations together. Box-counting dimensions give us the same approximations as Hausdorff dimensions; however, it can also be used to find those of more advanced fractals which may not be perfectly self-similar. Although still self-similar, take the von Koch Curve.

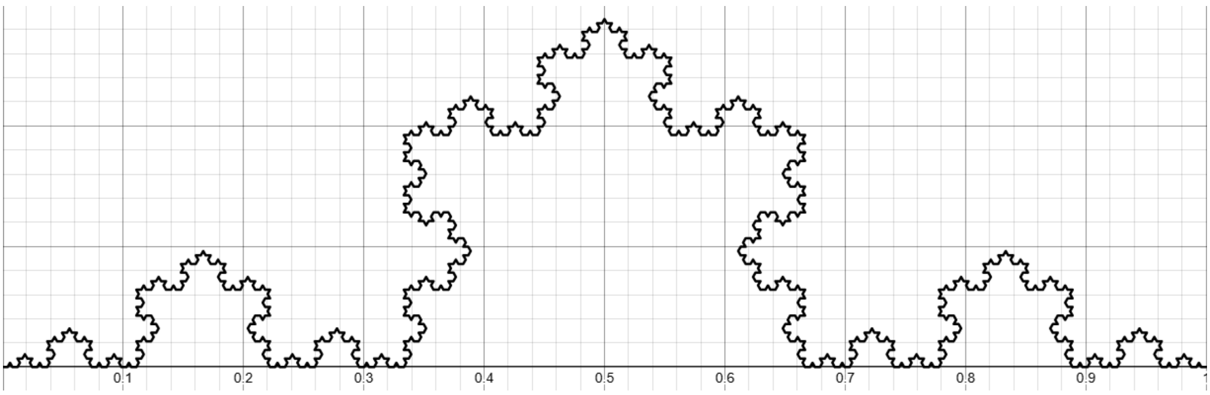

Figure 3: von Koch Curve.

To iterate the von Koch Curve, we split an initial line segment into 3 equal segments. We replace the central line with 2 of the same length to form an equilateral triangle. We repeat this on each remaining segment infinitely to construct our curve as shown above.

To estimate the box-counting dimension of this set, we first consider the smallest number of sets of diameters δ, such that $\delta > 0$, that may construct our curve. We shall coin this $N_\delta(K)$, where K is the set of our curve. The box-counting dimension represents this number's growth as $\delta \to 0$. As this increase is consistently proportional, we say that $N_\delta(K)$ follows a power law such that

$$N_\delta(K) \simeq c\delta^{-s},$$

where c is a positive constant, $\simeq$ represents errors disappearing as $\delta \to 0$, and where s is the box-counting dimension. We may once again take natural logs.

$$\ln N_\delta(K) \simeq \ln c - s \ln \delta$$

$$-s \ln \delta \simeq \ln N_\delta(K) - \ln c$$

$$s \simeq \frac{\ln N_\delta(K)}{-\ln \delta} + \frac{\ln c}{\ln \delta}$$

Thus, as $\delta \to 0$,

$$s = \lim_{\delta \to 0} \frac{\ln N_\delta(K)}{-\ln \delta}$$

This equation is rather similar to that of the equation for Hausdorff dimension, with the only addition being that it accounts for lower levels of self-similarity which may help us to determine the lengths of coastlines.

For our new curve, let $N_\delta(K) = 4$, as there are 4 initial segments needed for our iteration, and let $\delta = \frac{1}{3}$ for a consistent downscale ratio, with box-dimension $= D_B(K)$.

$$D_B(K) = \frac{\ln N_\delta(K)}{\ln \frac{1}{\delta}}$$

$$D_B(K) = \frac{\ln 4}{\ln 3}$$

$$\Longrightarrow D_B(K) = 1.2618595071429 \ldots \approx 1.262\ (3\ d.p)$$

As the floor of this value does not exceed 2, and due to the set belonging to 2-dimensional space, we know that this set has an *infinite* Lebesgue measure of length. This is because, as we are constantly adding line segments, the length never decreases. This is quite similar to the coastline, as will be proved.

6 <u>Conclusion</u>

Now that we have all the tools necessary to compute the box-counting dimension of a coastline, we can finally move onto understanding why such a formation has infinite length. Take a section of a coastline such that it is clear enough to lay out on an x-y plane. If we overlay a grid of squares each of side length δ and reduce δ to the smallest possible value that allows us to map the shape of the coastline - counting the number of squares at each decrement of δ - we can calculate the dimension using our formula.

Although we do not have specific figures for this, it is now no longer difficult to imagine that, after our calculation, we are left with a value between 1 and 2 – roughly 1.25 for the UK - as it may not exceed its 2-dimensional, x-y plane. In a technical sense, the coastline has infinite Lebesgue measure of length as shown with the von Koch Curve, a similar shape to that of the coastline, therefore, we may conclude that the length of the coastline is **<u>infinite</u>**, as we hoped to prove.

It is phenomena like these that make mathematics such an intrinsically beautiful concept, as, no matter how much you may attempt to suppress it, mathematics appears in all of our existence - decision making, weather patterns, coastlines - and I hope it is discoveries like these that encourage you, the reader, to continue exploring our mathematical world.