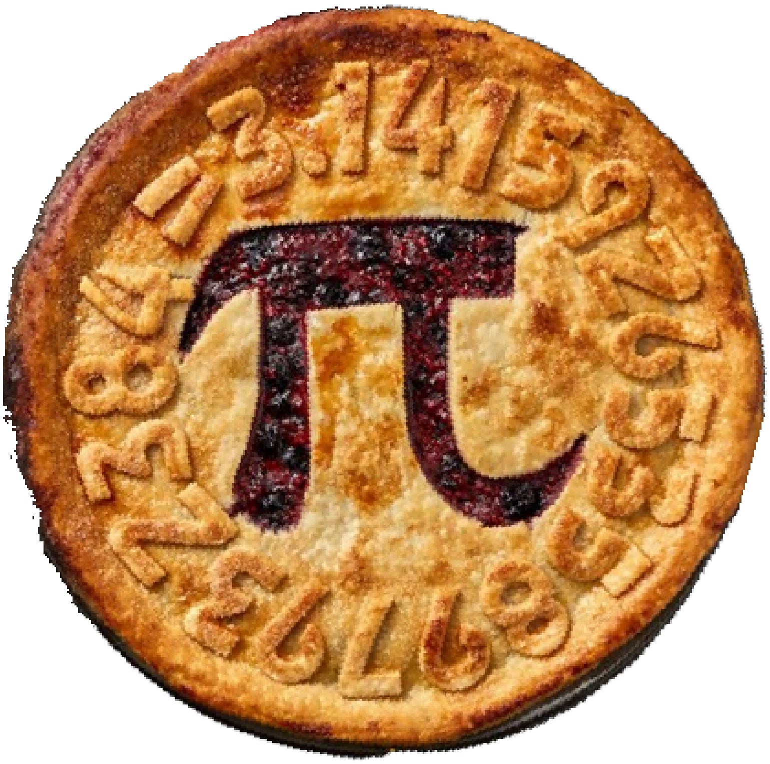
Pi explained by Pie

(The recipe for pi)



“Probably no symbol in mathematics has evoked as much mystery, romanticism, misconception and human interest as the number pi”

~William L. Schaaf, *Nature and History of Pi*

“I like pie. You like pie too?”

~Barack Obama

Pi. It’s everywhere; tattooed on mathematician’s wrists, printed on teacher's mugs. My brother and I even have duels to test how many digits we can commit to memory. But what is it and where did it come from?

Circumference (the Crust)
Here we have a tasty apple pie:

If we cut off the crust of the pie like so:

Then make the crust into one straight line:

We can then put this in terms of pie width:

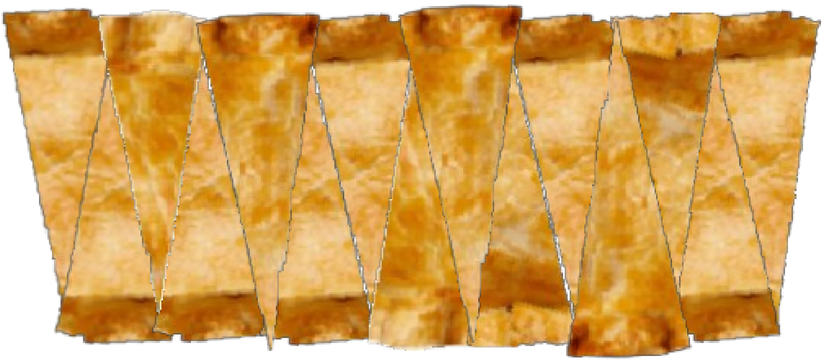
Width of one pie x total pie width = Length of crust

As we can see the crust covers 3 and a bit pie width with the last bit being roughly 0.14159 x one pie width so the total pie length is 3.14159 which we modal as the constant π. In this context the crust is the circumference and the length of a pie is the diameter. So:

Diameter x π = Circumference

Area in terms of pi(e)
To find the area we cut the pie into very small triangles:

We then rearrange these triangles to form a rectangle:



As we can see the length of the rectangle is half of the crust and the perpendicular height is half of the width of the pie meaning:

$$(0.5 \times \text{crust length}) \times (0.5 \times \text{width of pie}) = \text{area pie}$$

Crust length is the circumference which we have just discovered is Diameter x π and width of pie is the diameter so half the width of pie is the radius meaning:

$$\text{radius} \times \pi \times \text{radius} = \text{area}$$

Which simplifies to:

$$\pi r^2 = \text{area}$$

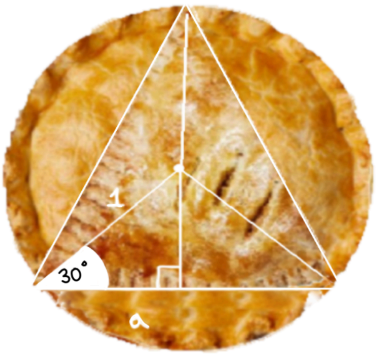
Here I use a rounded version of pi (3.14159). However, pi is actually an irrational number meaning it is infinitely long. But how can our added nubbin of pie be infinitely long?

Archimedes' Method (Pasties and cherry pie)

People have been making approximations of pi for over 4000 years now. Archimedes was the first to come up with an algorithmic approach.

Here we have a pasty on top of a pie:

We can easily work out the perimeter of a pasty as it has straight lines. This pasty is a similar size to the pie so we can use it to find a rough estimate of the pie's diameter to circumference ratio (pi). Assume the width of the pie is 2 (radius is 1). Consider the pasty on top assuming all sides are equal and therefore each angle is 60^o:



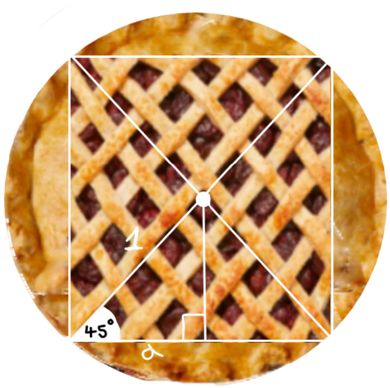
Using trigonometry we find a= $\sin 30^o$= √3/2. The perimeter would be 6a so perimeter = 3√3. The ratio of perimeter/ diameter for the triangle would be 3√3/2 ~ 2.6

Archimedes refined this method further as it wasn't very accurate at all.

Here we have a cherry pie on top of a round apple pie on top of another (bigger) cherry pie:

Let's again assume the width of the apple pie is 2 meaning the radius is 1.

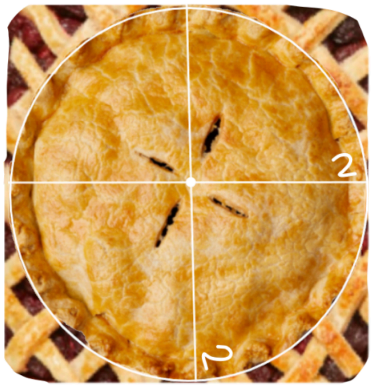
Consider the top cherry pie:



Using trigonometry we find a= √2/2. The perimeter would be 8a so perimeter= 4√2

The ratio of perimeter/ diameter for the square would be 2√2 ~ 2.8

Consider the bottom cherry pie:



As the diameter is 2 the length and base of the rectangle would also be 2. Therefore the perimeter is 4 x 2= 8. The ratio of diameter to perimeter is 8/2 = 4

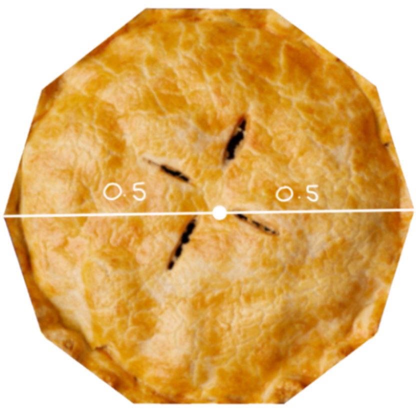
From these two calculations, Archimedes determined that pi must be between 2.8 and 4

Archimedes didn't actually use the following equation he continued bisecting polygons as calculators didn't exist but I will use it to delineate his findings:

Perimeter/diameter = n x sin(180/n)

Where n = the number of sides
And the diameter = 1
As we increase the number of sides (as we did above) this is n increasing.

So let's take a pie with 10 sides which looks something like this:



Perimeter/diameter = 10 x sin(18) ~ 3.0901699…
As we keep increasing the number of sides:

100 x sin(180/100) ~ 3.1410759…
1000 x sin(180/100) ~ 3.1415874…
10000 x sin(180/10000) ~ 3.1415926…

We get closer and closer to the known value of pi.

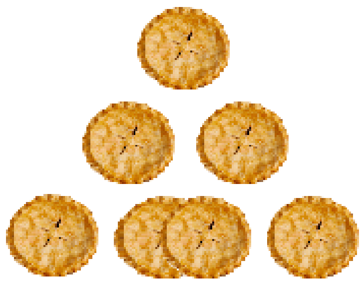
This makes sense as a circle is just a polygon with infinitely many sides. Therefore, as n tends towards infinity the perimeter/diameter tends to pi. The Dutch mathematician, Ludolph van Ceulen, determined, to a high accuracy, the perimeter of a polygon with 2^{62} sides which is just over 4 quintillion sides! This gave us only 35 correct decimal places of pi (which were later inscribed on his tombstone) but took him a total of 25 years. This method was tediously long so on the turn of the 18th century Isaac Newton came along and took a crack at it.

Isaac Newton and Pascal's Pie Stack (Pascal’s Triangle)

If we have a pie:

Then we add two more pies below it:

Then sum the number of these pies together and add these below it:



If we continue to do this we get this triangular shape:

We call this Pascal’s triangle and it’s been around for centuries. But Newton came up with a way to work out how many pies are in each group on each row. This is called the binomial theorem and it looks like this:

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!} \cdot x^2 + \frac{n(n-1)(n-2)}{3!} \cdot x^3 \ldots$$

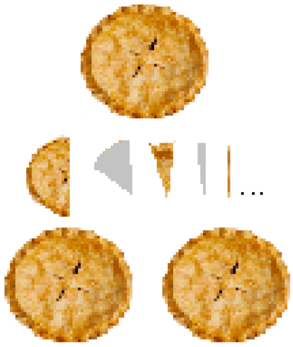
Where n represents the rows of pies. So, if we substituted n=0 we would get the top row and we would get one as there is only one pie. If we substituted n=2 we would get:

$$(1+x)^2 = 1 + 2x + 1x^2$$

As we can see, we end up with the co-efficients; 1, 2, 1. This gives us the number of pies on the second row:

The x cubed term and all following terms are eliminated since each of the coefficients is multiplied by the bracket (n-2) so when n is 2 the bracket = 0. This means the sequence is finite.

Originally, the sequence was only deemed applicable for the natural numbers. Newton ignored this rule and played around with both fractions and negatives instead.



Above we have what happens when you substitute n=0.5 (Row 0.5) where the grey pieces of pie are being taken away from the normal ones. We end up with infinite pieces of pie adding eventually to make 2. I'm using n=0.5 as the equation of a unit circle is:

$$x^2 + y^2 = 1$$

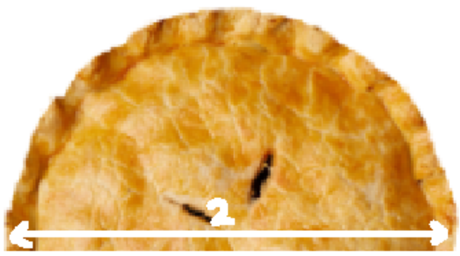
Which we can rearrange to be:

$$y = \left(1 - x^2\right)^{0.5}$$

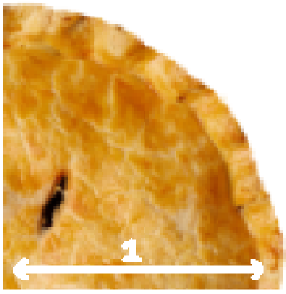
This looks a lot like the format for our binomial expansion therefore we expand it with a coefficient of negative one to get:

$$\left(1 - x^2\right)^{0.5} = 1 - 0.5x^2 - 0.125x^4 \ ...$$

Luckily for Newton, he had just invented calculus so he applied what he called the "theory of fluxions" which is now famously known as integration. Integrating the function gives us the area of a pie which we determined on page three was πr^2. Since we square rooted the function earlier we only have the positive terms so the function looks like this:



The radius is one so the area of the whole pie is π. Here we have half of that so integrating the function gives us 0.5π. Integrating between 0 and 1 would give us:



Which equals 0.25π. So if we integrate the function, set it equal to 0.25π we get:

$$\frac{1}{4}\pi = x - \frac{1}{2} \cdot \frac{x^3}{3} - \frac{1}{8} \cdot \frac{x^4}{4} - \frac{1}{16} \cdots$$

Substituting x=0 gives us 0
Substituting x= 1 gives us the following:

$$\frac{1}{4}\pi = 1 - \frac{1}{6} - \frac{1}{40} - \frac{1}{112} - \frac{5}{1152} \cdots$$

We can then rearrange to give us a value of pi to an arbitrarily high precision:

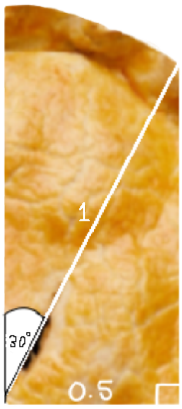
$$\pi = 4\left[1 - \frac{1}{6} - \frac{1}{40} - \frac{1}{112} - \frac{5}{1152} \cdots\right]$$

However, mathematicians are lazy and this would take a bit of time to complete to a high degree of accuracy so Newton decided to integrate from 0 to a half instead giving us this part of the pie:

But we can't just substitute 0.5 into the equation we have to discover what this piece of pie is in terms of pi:

By splitting the pie into a sector and a right angled triangle we can work out the angle of the sector using trigonometry:

Angle (degrees) of sector= 90 - arccos(0.5/1) = 30



We can then use this to work out the area of the sector using the equation:

$$\text{Area of sector} = x/360 \times \pi r^2$$

By substituting x= 30 we get the area of the sector to be π/12. We can then find the area of the triangle using the equation:

$$\text{Area triangle} = 0.5 \times a \times b \times \sin(C)$$

Substituting the values gives us:

$$\text{Area triangle} = 0.5 \times 1 \times 0.5 \times \sin(60) = \sqrt{3}/8$$

Therefore we can set the integral equal to π/12 + √3/8 (total area of segment) like so:

$$\frac{\pi}{12} + \frac{\sqrt{3}}{8} = x - \frac{1}{2} \cdot \frac{x^3}{3} - \frac{1}{8} \cdot \frac{x^5}{5} - \frac{1}{16} \cdot \frac{x^7}{7} \cdots$$

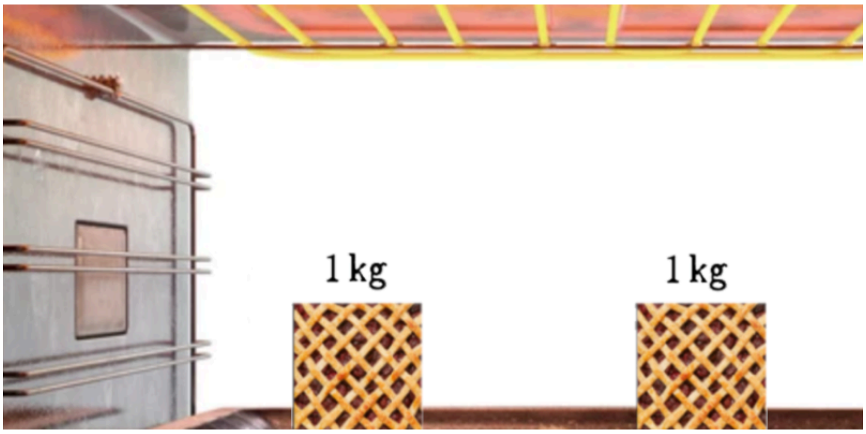
When we substitute x= 0.5 then rearrange we get:

$$\pi = 12\left[\frac{1}{2} - \frac{1}{2}\frac{1}{3}\left(\frac{1}{2}\right)^3 - \frac{1}{8}\frac{1}{5}\left(\frac{1}{2}\right)^5 - \frac{1}{16}\frac{1}{7}\left(\frac{1}{2}\right)^7 \cdots - \frac{\sqrt{3}}{8} \right]$$

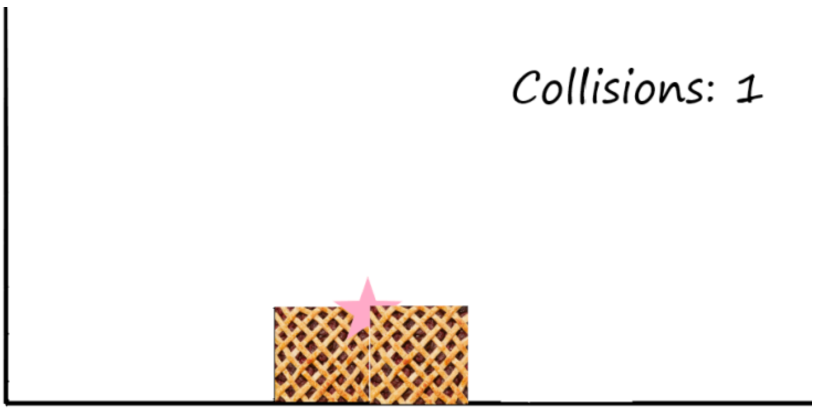
By evaluating the first five terms we get π= 3.14161. This is much easier than calculating sides of a polygon. The degree of accuracy that cost Van Ceulen 2^{26} sides of a polygon and 25 years of his life took Newton 50 terms in his series and a few days.

So far, I have given an explanation and brief history of pi but this little symbol creeps up in all fields of mathematics. The following is just one example of its many unlikely appearances here showing up in a physics experiment.

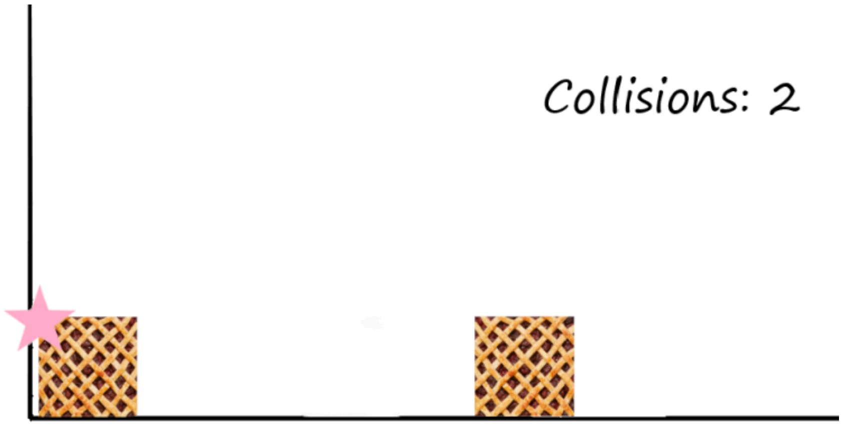
Consider two cherry pies in an (infinitely long) oven where the surface is smooth with 0 frictional forces and the same mass:



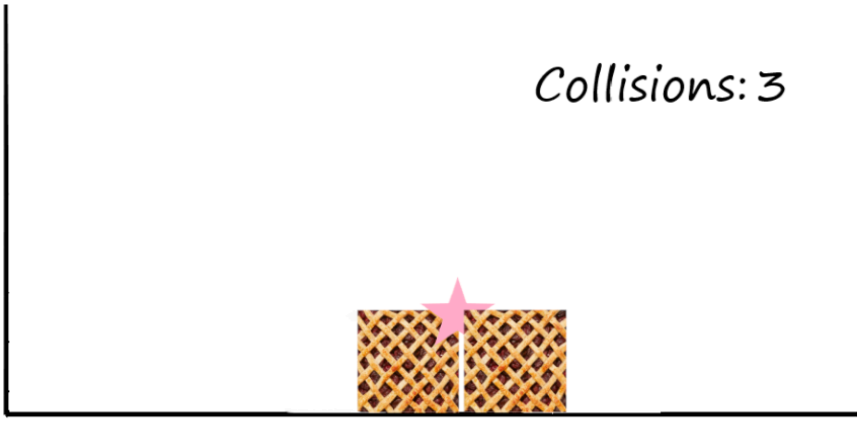
If the pie on the right moves towards the one on the left they will collide and it will transfer all its momentum.

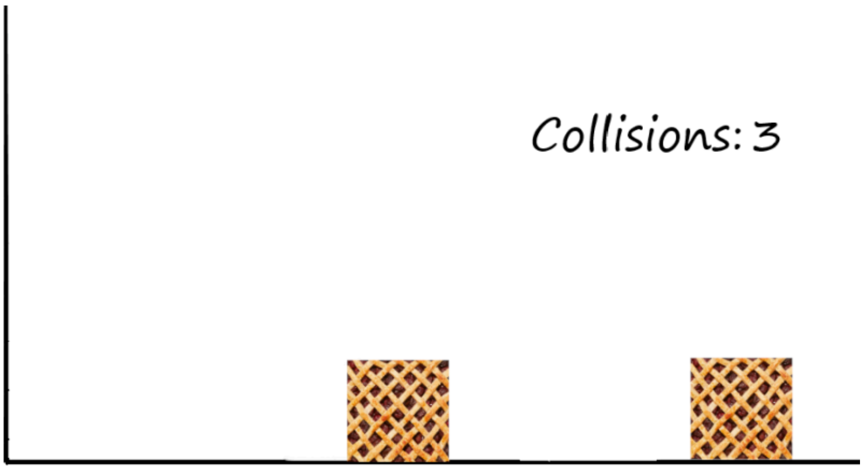


The pie will then hit the oven door.



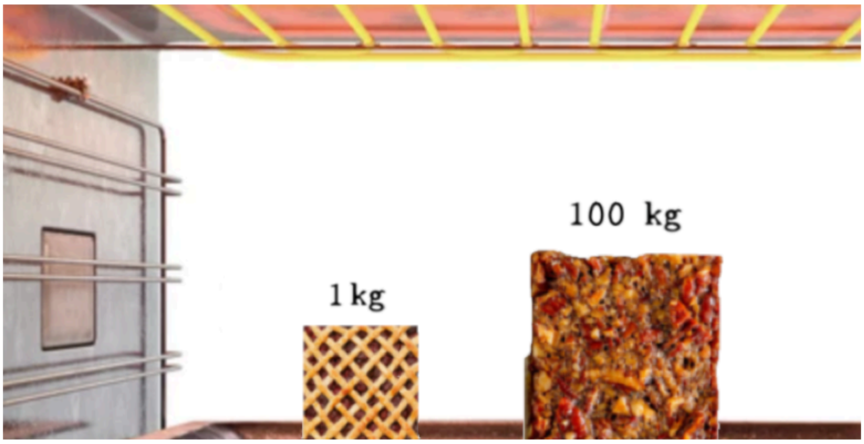
And then transfers its momentum back to the first which then travels away to infinity.



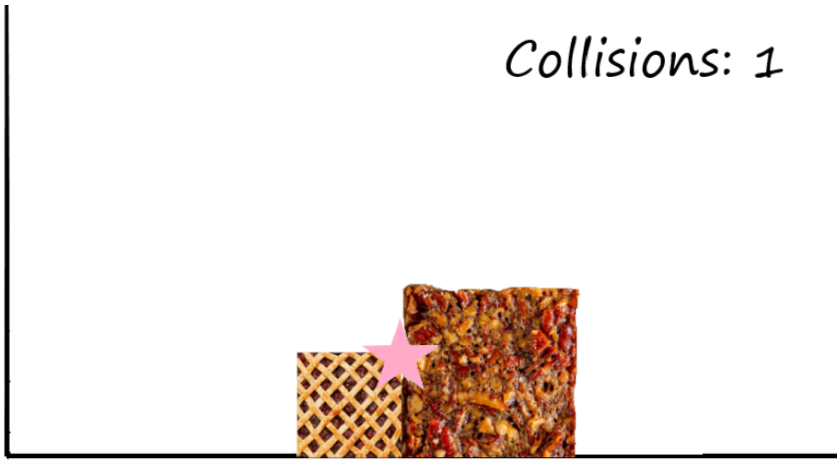


The total number of collisions is three.

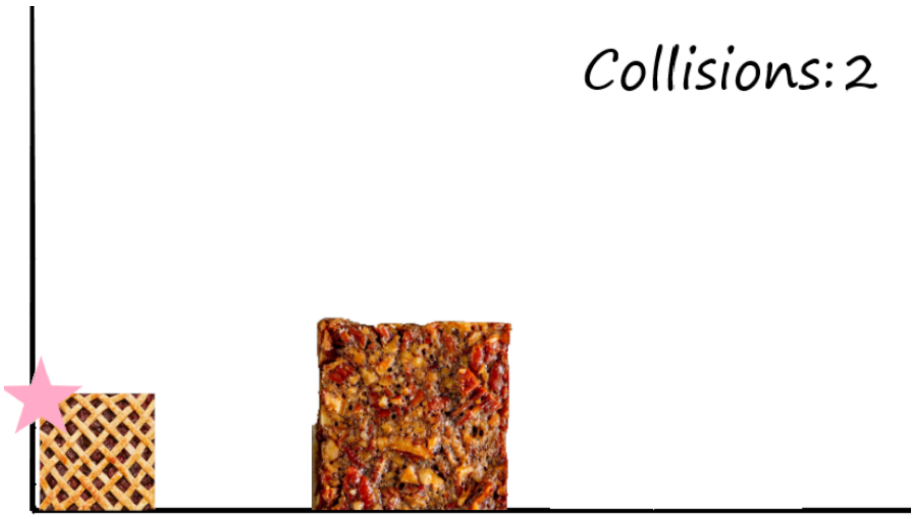
If we now consider a pie with a much bigger mass like this pecan pie that's travelling along the same oven surface:



These collide.

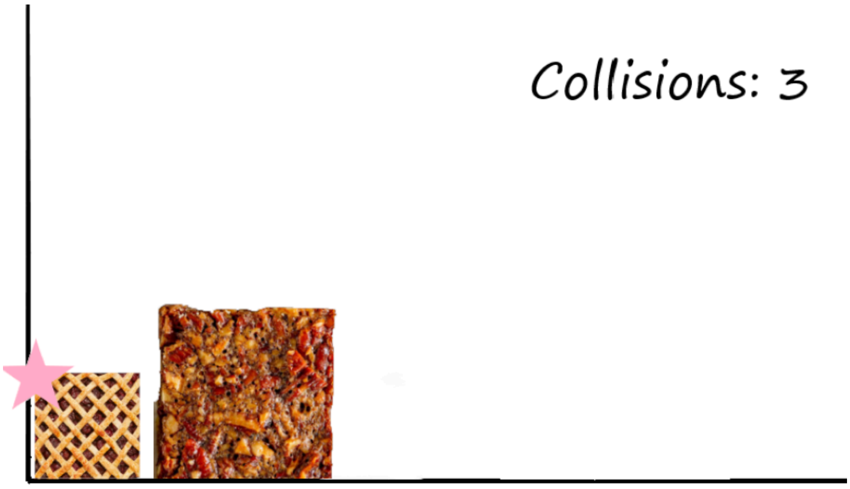


The cherry pie then hits the oven door just like our first pies did, however, not all of the momentum is transferred as the pecan pie has more mass so the pecan pie continues moving towards the cherry pie.

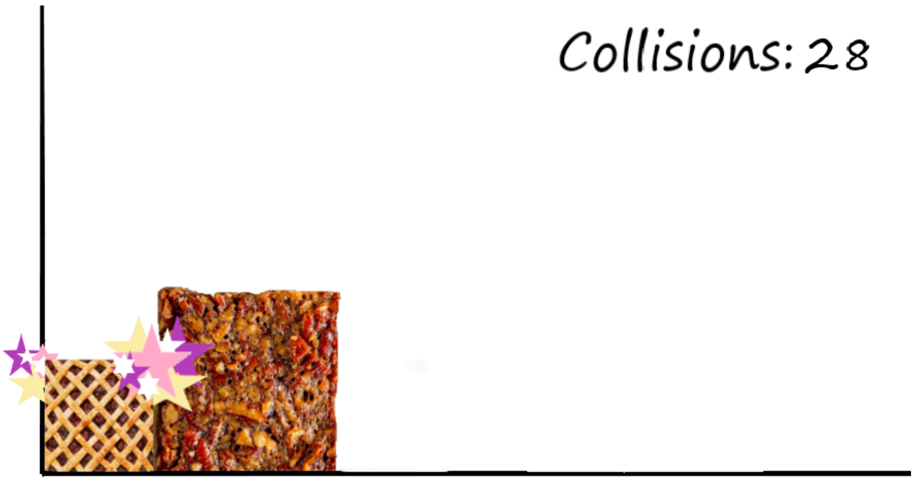


These then collide again:

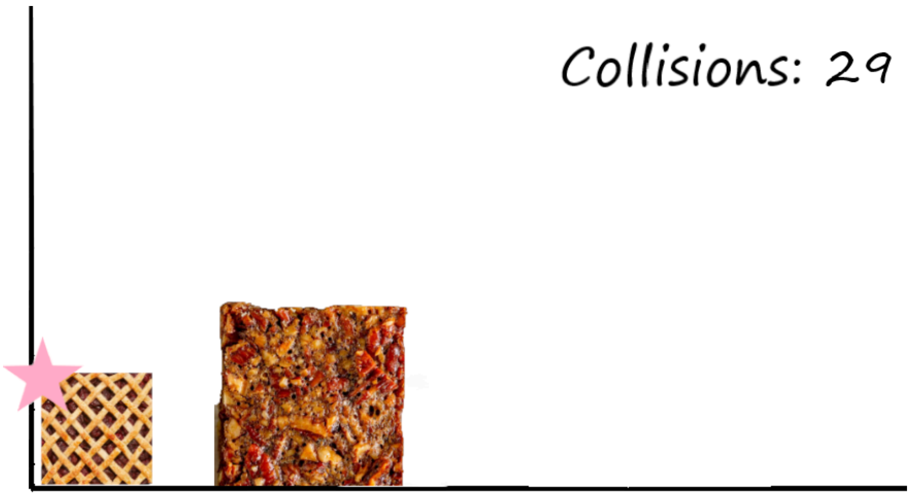
Then hit the wall again:

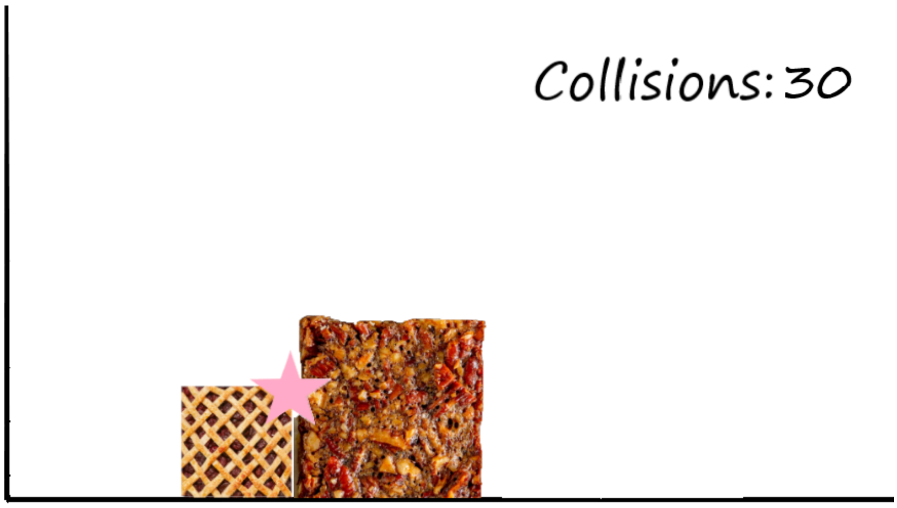


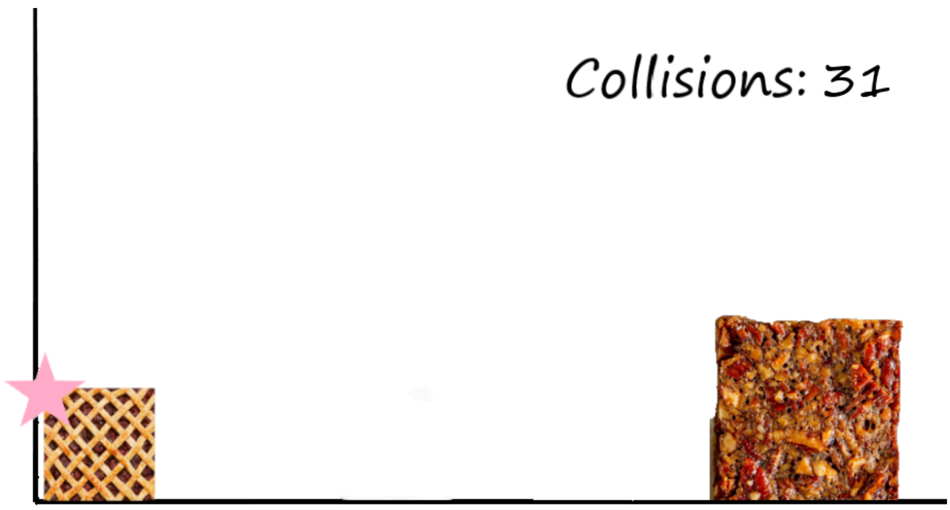
This continues happening until the pecan pie changes direction.



After a few more collisions the pecan pie is sent travelling to infinity.







As we can see the total number of collisions is 31. If we continue to increase the mass of the pecan pie by increasing powers of 100 (considering the mass of the cherry pie stays the same) then we get:

Weight of pecan pie (kg)	Number of collisions
1	3
100	31
10,000	314
1,000,000	3141
100,000,000	31415
10,000,000,000	314159

And so on. Have you spotted it yet? The numbers in the right hand column are the digits of pi!

This clearly isn't an effective way of computing pi as these are very unrealistic conditions. To obtain just 20 digits of pi you would need a pecan pie of mass 100^{20} times the mass of the cherry pie which would be 100 billion billion billion billion kg. This is ten times that of the supermassive black hole in the middle of the milky way.

We better stick to calculus.

The connection between the number of collisions and pi is ultimately unsolved but has something to do with the conservation of energy equation:

$$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = const.$$

Ultimately, pi pops up everywhere. Sometimes in mysterious, unsolved ways. It is simply the ratio of the circumference to the diameter but builds the foundations of mathematics. That's why I like pi. You like pi too?