

The Foundation of Combinatorics

Pigeonhole Principle

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The Birthday Paradox: An Intuitive Starting Point

Recall the familiar birthday paradox: it states that in a room with 23 people, there is around a 50% chance that two people share the same birthday.

Today I will be talking about a similar yet intuitive scenario: Imagine that you are in a room with 12 other people, what are the chances that at least 2 of you share a birth month?

The answer is 100%. At first this may seem surprising, since we did not calculate any probabilities how did we know for certain that at least 2 of them share the same birth month? How did we know that the result must be true. You might say, ‘there are 12 months and 13 people, therefore this forces an overlap and therefore at least 2 people will share the same birth month!’ Exactly! And this is also known as the **Pigeonhole Principle**. This illustrates how quickly probability grows — often in ways that feel counter-intuitive: a mere 23 people can raise the probability from 0 to over 50%! Unlike the birthday paradox, which is probabilistic, the birth month example guarantees this overlap with absolute certainty.

Daily Applications of the Pigeonhole Principle

This does not just apply to birth months, there are infinitely many scenarios that you could apply the Pigeonhole Principle with, or you may even apply it

unconsciously throughout your lives. For example in this technological society where everything is bounded by online accounts - where you have probably seen people trying to pay hundreds just to get a 4 letter username on a platform. This is known as the Pigeonhole Principle, there are only 456976 (or 26^4) 4 letter usernames that only contain lower case letters, so in a platform containing millions to billions of users, this scarcity becomes inevitable yet valuable.

Or perhaps since exam season is slowly creeping up, you may worry about your grades. But this also uses the **Pigeonhole Principle**, there are only 7 different grades (A*/A/B/C/D/E/U) at A-level. However, there are far more than 7 students that sit the exam, therefore this implies that at least 2 students must receive the same grade. Perhaps we could argue that these letters limit you by your labels, even though everyone's ability is different (we may come back to this point later), the grading system forcefully overlaps you and other students.

Now that you've seen the ubiquity of this principle, let's look at it from a philosophical perspective. Even though every student's exam script is different, there are people getting the same grade, or even the same marks. So this raises a question, '**In a finite world, can uniqueness truly exist?**', despite everyone being born differently, from an infinite perspective, it all seems to be guaranteed, from the month, to the day, to the minute, to the second, so are we truly unique? If the number of possible states in a system is finite, then repetition is not merely possible — it is unavoidable over an infinitely long scale.

And this is why I chose this topic, such an obvious principle, yet it raises lots of questions around society, individuals or even the laws of this universe.

Pigeonhole Principle in Mathematics

The Pigeonhole Principle is the foundation of modern mathematics. For example, it often underlies **Number Theory** - the study of numbers.

In **divisibility**, the principle often lies within the divisors, when dividing an integer n by another fixed integer, there are only finitely many remainders in the set $\{0, 1, 2, \dots, n-1\}$, so when we check this with $n+1$ integers, this implies that at least 2 of those integers share the same remainder, and that their dif-

ference is divisible by n , this is very similar to the **Euclidean Algorithm** - which is a cornerstone for Number Theory as this idea is often used everywhere, though that's a topic for another day. The Pigeonhole Principle often allows to use to solve difficult problems (often this is used with **orders**) - which links to more fields such as **Abstract Algebra** or **Group Theory**. The Pigeonhole Principle is a cornerstone of mathematics, as how it could be as trivial as a birthday problem to as complex as **Lagrange's Theorem** - where it states that the order of an element must divide the size of the entire group - it is this inevitability that forces structure into seemingly randomness.

Another example of the Pigeonhole Principle being a fundamentally powerful tool in problem solving was displayed in the infamous International Mathematical Olympiad. This is a problem from the IMO in 1972 and a proof that I've written using the Pigeonhole Principle.

Problem (IMO 1972/1)

Prove that from a set of ten distinct two-digit integers, it is always possible to find two disjoint subsets whose members have the same sum.

Solution

Firstly, there are $2^{10} - 1 = 1023$ non-empty subsets.

The total sum S of any of these subsets must be $\leq 90 + 91 + 92 + \cdots + 99 < 99 \cdot 10 = 990$.

By the Pigeonhole Principle, as $990 < 1023$, there must exist two distinct subsets with the same sum.

Define these two sets as A and B . To show they can be made disjoint, remove any common elements from both subsets. Removing the same numbers from both keeps the sums equal. The resulting sets are disjoint and still have equal sum.

Moreover, they cannot both be empty, since that would imply the original subsets were identical.

Therefore we obtain two disjoint subsets with the same sum, and we are done! $\square$

It should be evident how important the Pigeonhole Principle is to modern mathematics and how it will continue to be.

Philosophical Perspective

Now that you've seen the ubiquity of this principle, let's also look at it from a philosophical perspective. Even though everyone's exam script is different, there are people getting the same grade, or even the same marks. So this raises many questions, **'In a finite world, can uniqueness truly exist?'**, despite everyone being born differently, from a mathematical perspective, it all seems to be guaranteed, from the month, to the day, to the minute, to the second, so are we truly unique? If the number of possible states in a system is finite, then repetition is not merely possible — it is unavoidable over an infinitely long timescale.

That's not just it, there are far more applications for the Pigeonhole Principle. We often say that everyone is unique. In many ways I agree, our choices, our experiences and our perspectives are all deeply personal and different. However, mathematics tells us that although the variation and permutation of genes are astronomically large, in principle this overlap is inevitable. Uniqueness may not lie in a single trait — intelligence, athleticism, or appearance — but in the intricate combination of countless traits interacting together. Is uniqueness a trait, or is it a temporary lie before structure forces inevitability?

This does not diminish individuality, instead the opposite. It demonstrates that we are not bounded by an isolated trait, but rather a combination of numerous traits, that causes this inevitability to go to great lengths. Although chimpanzees are 99% similar from us *Homo sapiens*, we can definitely tell the difference between us and them. The vastness of these combinations pushes repetition far beyond our practical experience, even if mathematics tells us it cannot be ruled out in theory.

The Pigeonhole Principle gives us an opportunity to see that even in the most complex systems that we may not understand yet, we already know that overlaps are inevitable and it is governed by structure.

Summary

What began from noticing a crucial and trivial feature of sharing birth months now reveal a hidden characteristic. In a room with 13 people, sharing birth months may feel like a coincidence, but the Pigeonhole Principle tells us this isn't random, but instead it is a system that structures all of mathematics - or

even our world in this case. It governs over divisibility, groups, orders, digital usernames and many more.

Perhaps this is the subtle brilliance of mathematics, it reminds us that in any finite space, overlap isn't just a coincidence, but a hidden structure that is shaping the world around us.