Points at Infinity
in a Cardboard Box

Hosna Mojahedi

1. Introduction

I first encountered Dobble as a child, the kind of game you play absent mindedly. It was only recently that I picked it up again and felt something shift. The rules hadn't changed. But now they bothered me.

The rules are simple: each card shows a handful of symbols, and any two cards share exactly one. **Every pair. Always.**

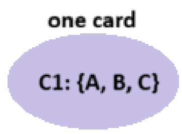
Try it yourself. It becomes unsettling very quickly. It isn’t approximate. It’s exact. And it raises an obvious question: how on earth do you build something like that?

First, the constraints: each card has k symbols, and each symbol appears equally often, to stay fair. Without this, the whole thing collapses. So let's try to construct a deck from scratch and see what we find!

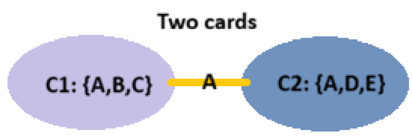
Start small. k=2: works trivially (try it yourself!). Let's make it harder.

k=3: I'll start placing them one at a time, and track what's happening using labeled graphs.

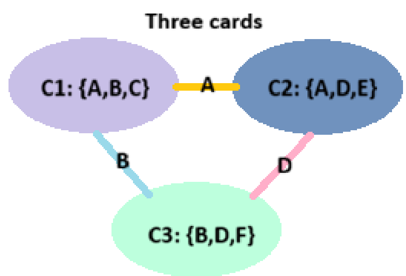
Card1:



Card2: Must share exactly one symbol with Card1:



Card3: Shares one symbol with Card1, and one with Card2.



I'll spare you Cards 4 through 7 because the process is the same. The graphs below tell the story. But I encourage you to try it! You'll find by Card 5 or 6, you have almost no choices left. The deck builds itself.

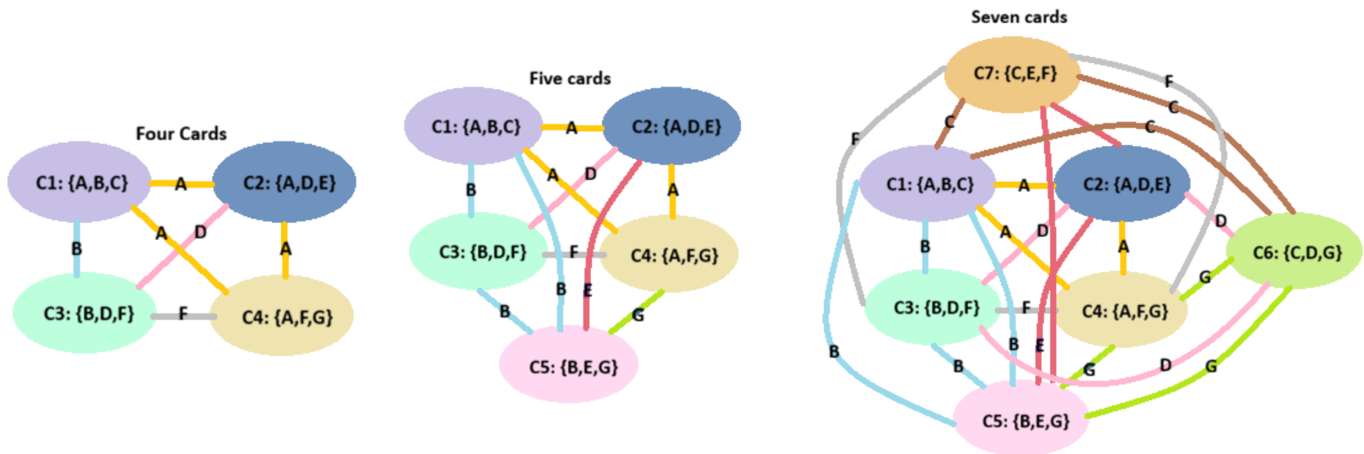


Figure1:Deck with k=3

Seven Cards, Seven Symbols

2. Follow The Numbers

Let's go again. k=4 this time.

It's the same process as before, just more constraints to juggle. But the answer is: it works, and the complete deck has 13 cards, drawn from 13 symbols.

Do you notice something? The number of cards always equals the number of symbols. That's not obvious, why should it? And the numbers 7, 13... let's stare at those for a second.

k	k-1	Deck size
3	2	7
4	3	13
8	7	57

A pattern is slowly emerging! If cards have k symbols, the deck seems to need $(k-1)^2 + (k-1)+1$ cards.

Let's check: for k=3, that gives $2^2+2+1 = 7$. For k=4, it gives $3^2+3+1 = 13$. What about k=8, like real Dobble? That gives $7^2+7+1 =$ **57**. The real deck has 57 symbols and 57 cards (55 after the manufacturer removes two for packaging). The formula holds.

But *why*? What do 7, 13, 21, 31, 57 have in common? Let's write out the values of k−1 that generate them:

For k=3 we had k−1=2, and everything fell neatly into place. For k=4, k−1=3, same story. Push further and the successes continued: k−1 = 4, 5, 7...

Then, just as consistently, the failures appeared: 6, 10, 12...
you run out of symbols before every pair of cards shares exactly one, no matter how you rearrange them.

At first glance this looks erratic. Why should 6 fail but 7 succeed? Why should 10 behave differently from 9?

But write the successful values in a different way:

2, 3, $4=2^2$, 5, 7, $8=2^3$, $9=3^2$, 11...

Now the pattern stops looking mysterious. The values that work are *exactly* the **prime powers**.

That's a much stronger claim than anything we've seen so far. So the question shifts. We're no longer asking how to build these decks. We are asking what's so special about prime powers that makes the construction possible.

To answer that, we have to leave combinatorics behind and look at something that, at first glance, has nothing to do with card games at all.

3. Geometry You Didn't See Coming

The problem with parallel lines

Take the real plane R^2. It has two things: points and lines. Pick any two distinct points and they determine a unique line. Pick any two distinct lines and they **almost** always meet in exactly one point. The exception is parallel lines, which don't meet at all. That word '*almost*' is mathematically irritating. The rule "any two lines meet in exactly one point" would be beautiful, clean, no exceptions. Parallel lines ruin this perfection. Mathematicians care about such elegance, so instead of tolerating the flaw, they set out to remove it.

The fix is elegant. For each direction in the plane, introduce a new **point at infinity** representing "where lines of that slope meet when extended far enough." Two parallel lines now share this point at infinity. Add one more line, the **line at infinity**, containing all these new direction-points. The result is the **real projective plane**: a geometry where every two distinct lines meet in exactly one point, and every two distinct points determine exactly one line. No exceptions.

This is beautiful, but it's an infinite geometry with infinitely many points and lines. For Dobble, we need something *finite*. What if we built the same construction, but replaced the real numbers with something smaller?

Arithmetic that wraps around

A **field** is a number system where you can add, subtract, multiply and divide freely, and the usual rules of arithmetic hold. The real numbers and rationals are fields. But there are also *finite* fields, with only finitely many elements.

The simplest example: take any prime p, and consider the integers {0, ..., p−1} with arithmetic done modulo p. In F_7 (the field with 7 elements), we have 3 + 5 = 1 (since 8 mod 7 = 1), and 3 × 4 = 5. Every nonzero element has a multiplicative inverse. In F_7, the inverse of 3 is 5 because 3 × 5 = 15 = 1 mod 7. It's ordinary arithmetic, just wrapped around at 7.

A fundamental theorem in field theory says that finite fields exist **if and only if** their size is a **prime power**. So there's no finite field of size 6. Hold that thought.

Building a finite projective plane

Fix a prime power q and take the vector space F_q^3. This space has q^3 elements in total. Now declare two nonzero triples **equivalent** if one is a scalar multiple of the other :
$(x, y, z) \sim (\lambda x, \lambda y, \lambda z)$. The **points** of our geometry are these equivalence classes. How many points are there? The number of nonzero triples is q^3-1. Each equivalence class has exactly $q-1$ members. So: Number of points = $(q^3-1) / (q-1) = q^2+q+1$

Lines are defined identically, equivalence classes of nonzero triples [a:b:c], containing all points satisfying $ax+by+cz=0$. By the same count: q^2+q+1 lines, each containing $q+1$ points.

And the crucial property:

> **Any two distinct points lie on exactly one line. Any two distinct lines meet in exactly one point.**

Let's see why, take two distinct points in PG(2,q). Each point is a whole *direction*, an equivalence class of nonzero vectors in F_q^3. So picking two distinct points means picking two nonzero vectors v and w where neither is a scalar multiple of the other.

Now consider all linear combinations av + bw, where a and b range over F_q. This sweeps out a two-dimensional subspace of F_q^3. Every nonzero vector in this plane is a direction, and therefore a point in our projective geometry. A two-dimensional subspace over F_q has q^2 vectors in total, of which q^2-1 are nonzero, and each equivalence class has $q-1$ elements, giving us exactly $(q^2-1)/(q-1) = q+1$ points. That's precisely the number of points per line we calculated earlier. So this plane through the origin is exactly what we mean by a line in PG(2,q).

Why is it unique? The span of v and w is the smallest subspace containing both, and there is only one such subspace. Two independent directions literally determine one plane through the origin, and therefore exactly one line.

The second part of the property follows by duality: points and lines in PG(2,q) are defined identically, just with roles swapped. The proof is the same argument, mirrored.

This geometry is called **PG(2,q)**, the projective plane over F_q.

Back to Dobble

Set q=7. The projective plane PG(2,7) has 57 points and 57 lines, 8 points per line, any two lines sharing exactly one point. Lines are cards, points are symbols. That's Dobble! 57 cards, 57 symbols, 8 per card. Every mysterious property of the game is now a theorem about PG(2,7).

The Fano plane

Before we scale up to the full 57-card Dobble deck, let's build the smallest possible version. Set q = 2: the field F_2 = {0,1} with arithmetic mod 2. One plus one equals zero. That's it, the entire field.

The resulting geometry PG(2,2) is called the **Fano plane** (Figure2)

By our formula, it has q^2+q+1 = **7 points** and **7 lines**, with $q+1$ = **3 points per line**, and every two lines sharing exactly one point. Seven cards. Seven symbols. Three symbols per card.

Look at **Figure2**. Seven lines. Each line contains three points.

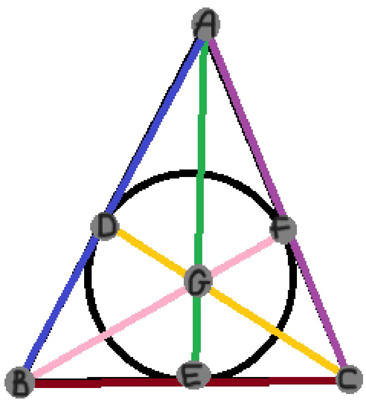


Figure2:Fano Plane

Pick any two lines from the Fano plane above. They share exactly one letter. This is precisely the 7 card deck we built in **Figure1** with the same symbols, the same structure, just drawn geometrically in **Figure2** instead of as a construction graph. The deck we assembled turns out to be the unique projective plane of order 2. We were doing projective geometry the whole time! The construction had no choice. The Fano plane is the only possible 7-card deck with 3 symbols per card. Once we impose the rules, the same number of symbols per card, and exactly one shared symbol between any pair the structure becomes completely rigid. We weren't discovering a solution, we were being forced into one.

But wait, could a projective plane of order 6 exist through some *other* construction, one that doesn't use finite fields at all?

4. Cracks in the Pattern

Does a projective plane of order 6 exist? The answer is no and the proof is a beautiful piece of number theory called the **Bruck-Ryser theorem**. It states:

> If a projective plane of order n exists, and n leaves a remainder of 1 or 2 when divided by 4, then n must be expressible as a sum of two perfect squares.

Now let's apply it to n = 6. Step one: does 6 leave remainder 1 or 2 when divided by 4? Yes. $6 = 1\times4+2$. The condition holds.

Step two: can 6 be written as a sum of two perfect squares? No.

So by Bruck-Ryser: **no projective plane of order 6 exists.** No 43-card Dobble deck. The construction was jamming because the thing we were trying to build is mathematically impossible.

The next non prime power is n = 10. Let's check Bruck-Ryser : 10 = 2×4 + 2. The condition applies. Can 10 be written as a sum of two squares? Yes: $10 = 1^2+3^2$. Bruck-Ryser has nothing to say. For all we knew in 1949, a projective plane of order 10 *might* exist.

Then in 1989, mathematicians announced the answer: **no projective plane of order 10 exists.** The proof was an elimination of every possible case, running for the equivalent of thousands of CPU hours. I don't know how to feel about this, a traditional proof is a sequence of logical steps that a human can follow and verify. Nobody understands *why* order 10 fails, not in the way you understand why order 6 fails. It fails because every single candidate was checked and none worked. Whether you find that satisfying is a matter of philosophy.

And then there is n=12. Bruck-Ryser doesn't apply. The search space for a computer proof is incomparably larger than n=10, we are nowhere near being able to check it computationally. And no theoretical obstruction has been found. So as of today we don't know whether a projective plane of order 12 exists (157-card Dobble deck, 13 symbols per card). It might be sitting out there, waiting to be constructed. It might be impossible. After 75 years of trying, mathematics cannot tell us which.

We started with a card game. We followed the constraints, noticed a pattern, and built a geometric framework that explained everything. And then we reached the edge of what is known, and it turns out the edge is only a few steps past something you can buy in a toy shop.