Primes:The Building Blocks of Numbers by Srishti Singh

The simplest definition of a prime number, p, is an integer>1 divisible only by 1 and p.

The reason this definition isn't completely accurate is because this doesn't take into account negative numbers. An alternative definition therefore is that prime numbers are any integers not equal to 0 such that if p=ab then a or b is a unit. But what is a unit? A unit is anything that has a multiplicative inverse (you can multiply it by another number and get 1, for example, for integers this is -1 and 1).

The Fundamental Theorem of Arithmetic states that if n≥1, n can be written as the product of primes in a unique way up to order. This means that n can be written as a unique multiplication of prime numbers, excluding the fact that if you write them in a different order (which would give the same number due to the commutative property of integers over multiplication),it is technically a different 'unique' product (1 is included in this because it is the product of the empty set).

But how do we know this for sure, well because unlike a lot of other things about primes, this has actually been proved.

Obviously, if n=1 (where it is a product of the empty set), or if n is a prime (where it is a product of the very prime that it is) this property is true.

But what about non-prime numbers>1.

Well, let's say that the first number that can't be written as a multiple of primes is n.

n=ab where a and b are both non-prime factors of n that are greater than 1 and less than n.

These numbers can be written as unique products of primes since n is the smallest number that can't be written as a product of primes and a,b are both smaller than n. So:

a=$p_1 \times p_2 \times \ldots..$

b=$q_1 \times q_2 \times \ldots\ldots..$

Therefore, n=$p_1 \times p_2 \times \ldots....q_1 \times q_2 \times \ldots..$ which are all prime numbers and so this is a contradiction and therefore there is no number which cannot be written as the multiple of prime numbers.

To prove uniqueness, we need to know that if a prime number,p, divides ab, then it must either divide a or b.

Suppose p doesn't divide a:
Then the highest common factor of a and p would be 1 since p is prime.

By Euclid, we know that ax+by=1 for some value of x and y.

Multiplying this entire expression by b we get, bax+bpy=b. bax is divisible by p because we have assumed ab to be divisible by p and so by extension, bax is also divisible by p. bpy is divisible by p because it has p as one of the factors. Therefore, since p divides bax+bpy and bax+bpy=b, p divides b. If p doesn't divide a it divides b, which is what we needed to prove to prove uniqueness.

Another proven fact about primes is that there are infinitely many primes. Euclid proved this by trying to build a number that couldn't be built from a finite list of primes. Whatever this

finite list is, if you multiply all the primes in it and add 1, you get a number not divisible by the product of this finite list and with the remainder 1. But since it had already been proven that all numbers could be written as a product of primes and given any finite list of primes we could always find a number whose product couldn't be found from that finite list, there had to be infinite primes.

Formula for the primes:

Mathematicians spent centuries trying to find a formula that could generate every prime number. There have been multiple attempts throughout mathematical history. For example, Mersenne's primes, which are primes with the structure 2^n -1, were thought to originally be generated if n equals the following numbers: 2,3,5,7,13,19,31,67,127,257. (It was later proven, after large computers started performing calculations that humans could only fathom doing that it is actually not true for n=67). Mathematicians believe that continuing Mersenne's list will produce an infinite number of primes but this has never been proven. Although, a point to note is that while it might produce an infinite number of primes as mathematicians believe it would, it does not produce every prime.

After multiple attempts of mathematicians trying to find a formula that generates all prime numbers, Gauss took a different road. He tried instead to calculate the number of primes between and 1 and a number, N.

He saw that the number of primes between 1 and 100, for example, was 25 and so there was a quarter chance of getting a prime number if you just took a guess.

He also observed that every time he multiplied N by 10, he had to add 2.3 to the ratio of the primes to all numbers. Logarithms turn multiplication into addition, and Gauss realised that in this particular case, he was looking at approximately a log of base e(because e is 2.71828….. which is roughly the same as 2.3)

Gauss estimated the probability of picking a prime between 1 and any number, N to be $\frac{1}{log(N)}$ and therefore the number of primes up to N would be the probability times N that is, $\frac{N}{log(N)}$.

The number of prime numbers between one and N are represented using π (as in a symbol and not the never ending decimal 3.14159…….)

Therefore, if $\pi(N+1)-\pi(N)$ is not 0, N+1 is a prime.

However, Gauss's approximation for $\pi(N)$ was found to gradually drift away from the number of primes as the value of N went up.

Legendre came up with an improvement to this approximation by changing the equation to $\frac{N}{log(N)-1.08366}$ but mathematicians weren't entirely convinced because of the presence of the 'ugly' number 1.08366. As Hardy said(even though he came into the mathematical picture much after this formula was suggested by Legendre), "There is no permanent place in the world for ugly mathematics."

Gauss came back into the picture and said that the number of primes was actually $\frac{1}{log(2)} + \frac{1}{log(2)} + \ldots\ldots + \frac{1}{log(N)}$. He even went as far as to creating the natural logarithm to estimate the number of primes and this was stunningly accurate. Primes estimated up to 3000000

using Li were only seven hundredth of 1 per cent off the mark, compared to Legendre's version which grew far less accurate after primes of 10000000.

The Zeta function

When fed with the number x, the zeta function equals $\sum_{n=1}^{\infty} \frac{1}{n^x}$ which in simpler terms just means $\frac{1}{1^x} + \frac{1}{2^z} + \ldots\ldots + \frac{1}{n^x}$.

This is equal to infinity for x=1, but for x=2, it is less. This is because now we're not adding together the reciprocals of all numbers up to N but only the reciprocals of the squares. Euler found this out to be $\frac{\pi^2}{6}$(this time π does indeed mean the never ending decimal 3.14159…..) Why is the zeta landscape relevant? Well because of the Riemann Hypothesis of course.

Riemann described the zeta function as a landscape of four dimensions, with two of the dimensions as the coordinates of imaginary numbers (numbers in the form of a+b**i** where **i** is an imaginary number that represents $\sqrt{-1}$) that are input into the function..

The third and fourth dimension was to record the output by the function.

In these imaginary landscapes, the zeros of the zeta function(imaginary numbers where the function outputs zero) are what is important.

Riemann had calculated a function R(N) which gave a fairly good count of the number of primes up to N. By adding to each of these guesses, the height of the wave of the graph of the zeta function above the zeros, he could correct the error produced by his function and find the exact number of primes up to N. Riemann had done what Legendre and Gauss could only approximate, found an exact number of primes up to a number N.

Riemann believed that every zero on this imaginary landscape would lie on the same line-with real coordinates of a $\frac{1}{2}$ and this belief has come to be known as the Riemann Hypothesis. He also realised that if the Riemann Hypothesis was true then Gauss was right in saying that his approximation for the number of primes up to N only got more accurate the larger the value of N. This was because the more easterly the zero lay on the landscape, the higher the error made by Gauss's approximation and so if the zeroes did indeed lie on the line from the Riemann Hypothesis then the error would be very small and so Gauss's conjecture would be correct. However, Gauss's conjecture could still be correct even if the Riemann Hypothesis was found to be false but not vice versa.

Hardy proved that there was an infinite number of zeros on the line from Riemann's Hypothesis but couldn't prove that these amounted to even a fraction of the total number of zeros. Just because there are an infinite number of zeros on the line doesn't mean that there can't be an infinite number of zeros elsewhere on the landscape.

Theory of probability and randomness of primes

From the theory of probability, it is known that for something to be considered fair, the 'error' produced by carrying out that calculation is expected to be in the order of $\sqrt{N}$. However, if something is biased than the error produced would be consistently higher than $\sqrt{N}$. Gauss's prediction of the number of primes up to N was an estimation and had errors, the question was whether these errors were random or had a bias? Every zero with real coordinates $\frac{1}{2}$

would produce an error that is the $\sqrt{N}$. Any further east and the zeros would produce a bigger error. So if the Riemann Hypothesis was proven to be true, the error margin produced would be in line with the prime numbers appearing randomly with no bias.

Riemann's Hypothesis and …..physics?

A mathematician named Montgomery looked for the gaps between the zeros in the zeta function based on the fact that randomness tends to lead to clusters (which is why buses always seem to come in three). When he mentioned the distribution he had found to a physicist Dyson, he said 'that's just the same as the behaviour of the different pairs of eigenvalues of random Hermitian matrices'. Behind all the physics jargon, an idea started to bloom in the mathematical world after this instance that perhaps it was the spacing between quantum energy levels that model the distribution of the zeros in the zeta landscape and are what is needed to help prove the Riemann Hypothesis.

Physicists believe that the reason the Riemann Hypothesis is true is because the zeros on the landscape will probably turn out to be frequencies of some mathematical drums. If the zeros don't lie on the line from Riemann's Hypothesis, these frequencies will be negative and this isn't possible by the rules of quantum physics. While there is data suggesting that the distribution of these mathematical drums is indeed similar to the distribution of the zeroes in the Riemann Hypothesis, this hasn't been proven yet and so the proof of the Riemann Hypothesis still remains out of reach.

Prime numbers have a long and varied history, not all of which I have even covered in this short essay for such a long subject. They have remained elusive for centuries and what seems like a relatively simple topic at first, the building blocks of maths, have managed to confuse minds as great as Euler. The Riemann Hypothesis still remains unproved but with all the new evidence considered, mathematicians are more optimistic than ever that it is actually true. It is hoped that hidden underneath the hypothetical proof of the Riemann Hypothesis, there is some hidden knowledge that will bring us closer to understanding the very building blocks, the so-called atoms, of the mathematical world.