Proof

by Harry Daniels

In many fields, the notion of proof is immediately associated with evidence. It could be said that evidence is an indication or clue that something is true. I imagine gathering evidence as the job of a detective, not at all that of a mathematician. This is where maths deviates from science; evidence alone is not enough. We do not focus on a single example, instead considering *every* example and applying our pre-existing knowledge - our axioms. With watertight proofs, we can have complete peace of mind (unless of course the mathematician is lying!)

'Exhausting' proof

Our first thought when trying to prove a statement for all outcomes is to find an outcome that works. Unfortunately, this is where some mathematicians stop; it works here therefore it must work every time. Nice try, but we still have some work to do!

So we go about proving it for every outcome.

Let's try to prove the number 2083 is prime. Prime means a number is only divisible by one and itself. Hence I'll need to divide 2083 by numbers smaller than it to see if I get an integer result. Using a systematic approach, to avoid missing any numbers, I'll start from the bottom and work my way up.

2083 / 2 = 1041.5 | 2083 / 3 = 643.333… | 2083/4 = 520.75 | 2083/5 = 416.6 …

I haven't even done five divisions and this is too much for me; it's exhausting stuff! This is why we have computers after all. Exhausting the possibilities as well as myself, this will be a 'Proof by Exhaustion.' And the computer tells me this is a prime number, as it has tried all possible numbers below it. Wikipedia tells me it's the 314th prime (maths reference.)

Since our understanding and mastery of computers has improved, we have been able to prove many other problems using exhaustion such as the Four Colour Theorem, a theorem that states at most four colours are needed to colour regions in a map such that no two adjacent regions share a colour, where computer technology tried 1936 configurations, much too many for a human to want to complete, especially in a systematic way.

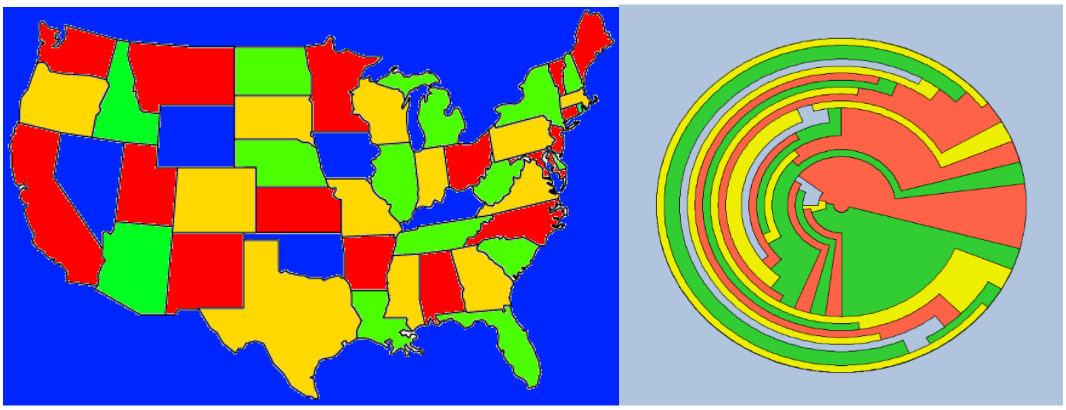

Figure 1 – Four colours are enough

Direct proof

'Proof by exhaustion' is an example of a direct proof; this is a proof where we are told the outcome and we follow some steps to show that the outcome will happen. It is the classic 'If a, then b' where we take a to be true and show that b must also be true.

For example, we can show that in "Conway's Game of Life", certain patterns live forever given the set of rules shown below.

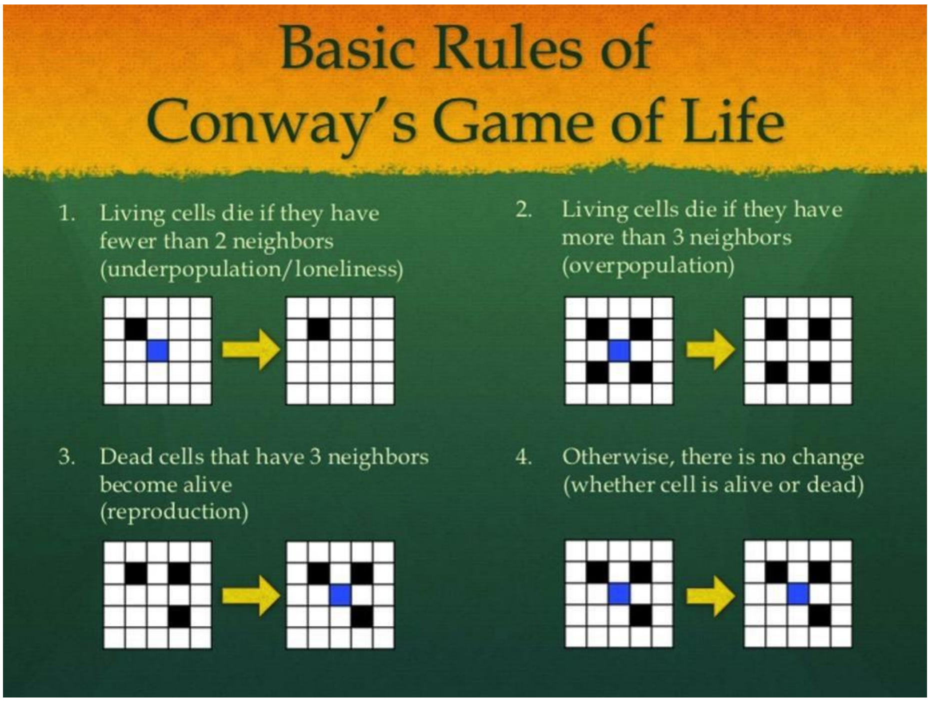

Figure 2 – The rules of Conway's Game of Life

Here, we can show the T-Tetromino, a pattern seen in the game Tetris, results in an oscillator, a pattern that will continue flipping between two states forever. We are proving that 'If I draw a T-Tetromino, it will live forever.'

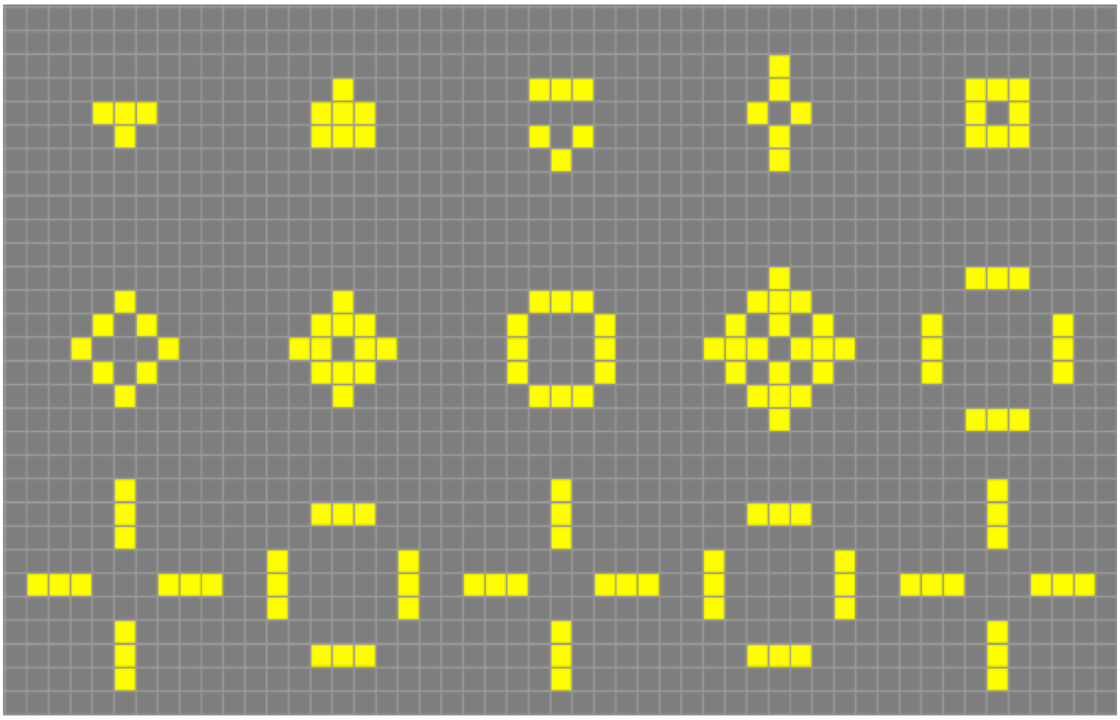

Figure 3 – The evolution of the T-Tetromino

Another form of direct proof, this time proving something is false instead of true (prove that something *does not always* apply) involves finding one single example where the condition is not upheld. While 'Proof by Exhaustion' requires consideration of every possible option to show absolute proof, a '(Dis)proof by Counterexample' is enough to disprove a whole conjecture. This further demonstrates the fact that strong evidence does not suffice and mathematical proofs must be beyond all doubt for any case possible.

This method was used in the infancy of computing to brute-force disprove Euler's Sum of Powers Conjecture, which states that a power cannot be expressed as the sum of other powers with the same exponent as the power where the number of powers added is less than the exponent and the exponent is greater than one. In fact, the publication of this counterexample found by Lander and Parkin was just two sentences long, demonstrating the power of a counterexample.

COUNTEREXAMPLE TO EULER'S CONJECTURE ON SUMS OF LIKE POWERS

BY L. J. LANDER AND T. R. PARKIN

Communicated by J. D. Swift, June 27, 1966

A direct search on the CDC 6600 yielded

$$27^5 + 84^5 + 110^5 + 133^5 = 144^5$$

as the smallest instance in which four fifth powers sum to a fifth power. This is a counterexample to a conjecture by Euler [1] that at least n nth powers are required to sum to an nth power, $n > 2$.

REFERENCE

1. L. E. Dickson, *History of the theory of numbers*, Vol. 2, Chelsea, New York, 1952, p. 648.

Figure 4 – Counterexample to Euler's Conjecture

Indirect proof

Unlike direct proof techniques, indirect proofs assume part of a statement is false, rather than true, to establish that the other part of the statement would have to also be false.

For example, 'Proof by Contrapositive' imagines b is false and shows that a must also be false. This is fundamentally the same as 'If a, then b' and is used where the negation of a statement is easier to manipulate. One example of a contrapositive statement I like is 'If it is raining, I take my umbrella out with me; if I didn't take my umbrella with me, it wasn't raining.'

This may be confused with the converse of a statement which is 'if b, then a', which is not the same as its forward version and will not be used as a method of proof.

- In the Rugby World Cup, there were formerly five teams in a pool, meaning it was impossible for all teams to be playing at once.
- If a pool has five teams, at least one team is not playing.
- The associated contrapositive would be "if every team in the pool is playing, there must be an even number of teams."

- Assume every team is playing, knowing that every match contains two teams. If there are n matches then 2n teams are playing. Hence there must have been an even number of teams.
- As 5 is odd it can't be represented as 2n because n = 2.5 which is not an integer.
- Hence it is impossible for all 5 teams to be playing at once.

This method is a variation of the Pigeonhole Principle which says if there were more pigeonholes than pigeons, then there would be a hole with two or more pigeons. To apply this here, if there were two pigeonholes, for the two matches taking place, at least one of the pigeonholes would contain three or more teams.

Figure 5 – Five teams per pool

Clearly, a format with an even number of teams is preferable as you could have all teams playing at once, as implemented in the FIFA World Cup. It breeds more chaos as we see groups flipped on their heads in the space of ninety minutes. In 2022's Group E, goals going in changed the placements seven times on the final matchday.

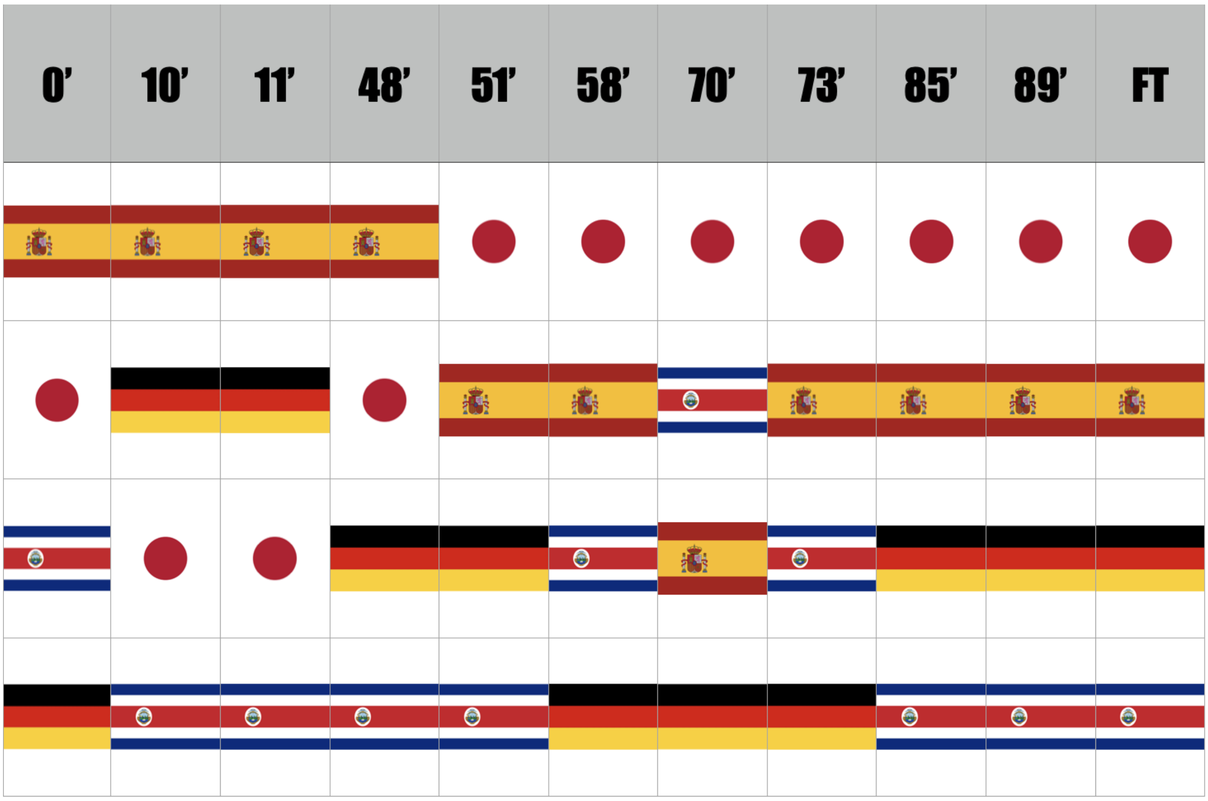

Figure 6 – Changes the placings over 90 minutes

Future editions of the Rugby World Cup will have the better four team format, such as the upcoming 2027 edition.

Figure 7 – Four teams per pool

Next, we should consider 'Proof by Contradiction', where we investigate what would happen if the statement was not true.

- We can take the triangle side length inequality, with the aim of proving it is true: "The sum of the lengths of two of the sides must exceed the length of the third".

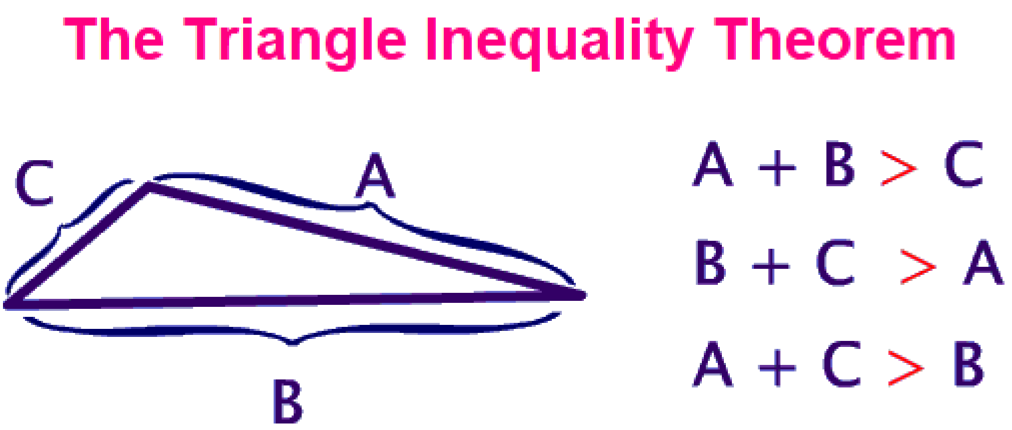

Figure 8 – Triangle Inequality Theorem

- To prove the statement, assume the opposite. Then see what would happen:

$$a + b > c$$

$$\text{assuming the opposite:}$$

$$a + b \leq c$$

- We use our knowledge that the shortest distance between two points is always a straight line between them.
- So we place the shorter two sides along the line of the longer side, from opposite ends.
- But either the lines don't touch, meaning we don't have three vertices, or else they *only just* touch meaning the figure has no area.

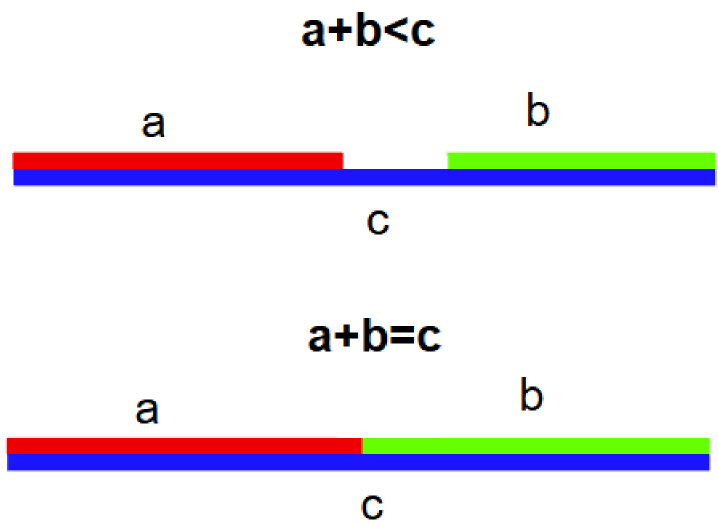

Figure 9 – Proving the Triangle Inequality Theorem

- Hence the side length inequality holds.

Proof by mathematical induction

Here we verify a statement is true for n=1, and demonstrate that if it's true for n=k it's also true for n=k+1. This gives us assurance that the statement must be true for all positive integer values of n. This technique is often used for summation formulae (where we add up sets of numbers) and to prove divisibility of an expression.

We will use proof by induction to prove the geometric series formula. A geometric series is the sum of the terms in a geometric sequence, where the next term is the previous term multiplied by a certain number. Such a formula could be used to find the total distance travelled by a bouncing ball after its first bounce.

nth term of a geometric sequence:

$$a_n = ar^{n-1} \qquad S_n = a \times \frac{1 - r^n}{1 - r}$$

- To prove the sum of the geometrical series formula, we use one as the base case.

- So we find the first term is a.
- This is also the sum at n=1 according to the summation formula.

Next we assume the same is true for n=k, writing the general formula as our result.

Next we use our earlier assumption to test if the same is true for n=k+1.

$$\frac{a(1-r^k)}{1-r} + ar^k \equiv \frac{a(1-r^{k+1})}{1-r}$$

$$\frac{a(1-r^k)}{1-r} + \frac{ar^k(1-r)}{1-r} = \frac{a - ar^k + ar^k - ar^{k+1}}{1-r}$$

$$= \frac{a - ar^{k+1}}{1-r}$$

$$= \frac{a(1-r^{k+1})}{1-r}$$

Therefore the summation formula is true for n=k+1 if it were shown to be true for n=k. Because it was true for n=1, it is true for all positive integers by mathematical induction.

Proof by construction

Proof by construction involves building a model that reaches the desired outcome. In some ways, this is similar to a counterexample as it requires just one example to demonstrate the possibility of something.

We can prove that the transitivity that applies to numbers is not always true for dice. For numbers, if a>b and b>c, then a>c. We might expect that if one dice beats another more than it loses and if the worse dice beats a third more than it loses, then the first dice should beat the third more than it loses. We can, paradoxically, construct three or more dice to show this is not true, just as we designed the game rock paper scissors so that selecting any item gives the player no inherent advantage.

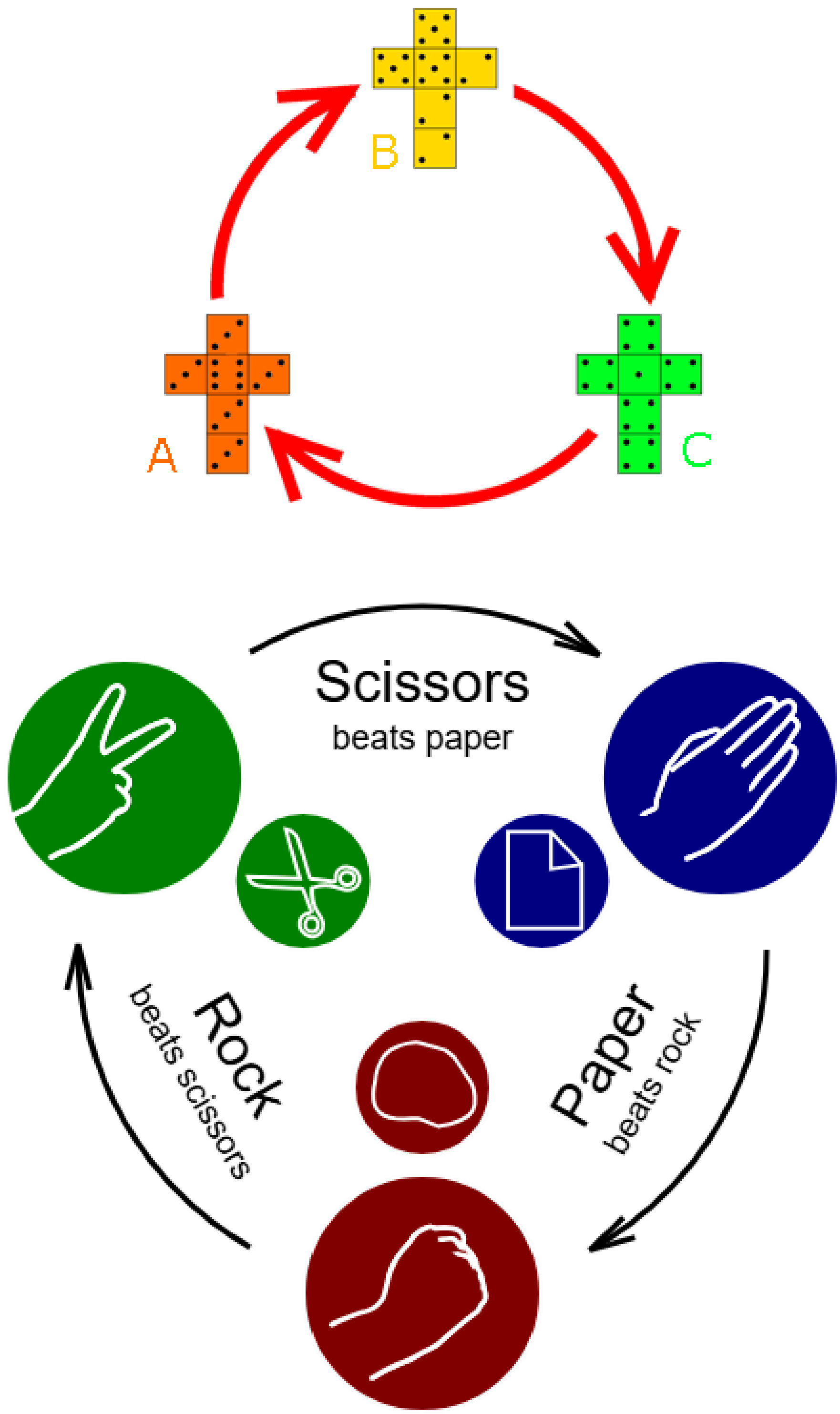

Figure 10 – Proving non-transitivity

Proof by invariant

Proof by invariant involves showing something is unchanged by a series of processes such as transformations.

We can look at invariants in the context of football league tables. One of the most common types of invariant proof involves parity of numbers, which is whether a number is even or odd. When a match happens, we either have a win (for one team) or a draw. When a draw happens, neither team's goal difference parity changes as they scored the same amount of goals as they conceded (n-n=0.) When a win by an odd number of goals happens, the parity of both teams changes since adding or subtracting an odd number changes the parity. When a win by an even number of goals happens, the parity is unchanged because both goal differences change by an even number. It is useful to consider parities as being like an on or off switch with only two possible states.

As the total number of goals scored equals the total number of goals conceded across the league, total goal difference will always be zero, even in cases where certain teams have incredibly high or low goal difference, like in the 2007/08 season, where Derby County achieved a record low 11 points. For instance, this invariance proves that it is impossible for all teams but one to have an even goal difference.

Pos	Team	Pld	W	D	L	GF	GA	GD	Pts	Qualification or relegation
1	Manchester United **(C)**	38	27	6	5	80	22	+58	**87**	Qualification for the Champions League group stage
2	Chelsea	38	25	10	3	65	26	+39	**85**	
3	Arsenal	38	24	11	3	74	31	+43	**83**	Qualification for the Champions League third qualifying round
4	Liverpool	38	21	13	4	67	28	+39	**76**	
5	Everton	38	19	8	11	55	33	+22	**65**	Qualification for the UEFA Cup first round
6	Aston Villa	38	16	12	10	71	51	+20	**60**	Qualification for the Intertoto Cup third round
7	Blackburn Rovers	38	15	13	10	50	48	+2	**58**	
8	Portsmouth	38	16	9	13	48	40	+8	**57**	Qualification for the UEFA Cup first round[a]
9	Manchester City	38	15	10	13	45	53	−8	**55**	Qualification for the UEFA Cup first qualifying round[b]
10	West Ham United	38	13	10	15	42	50	−8	**49**	
11	Tottenham Hotspur	38	11	13	14	66	61	+5	**46**	Qualification for the UEFA Cup first round[c]
12	Newcastle United	38	11	10	17	45	65	−20	**43**	
13	Middlesbrough	38	10	12	16	43	53	−10	**42**	
14	Wigan Athletic	38	10	10	18	34	51	−17	**40**	
15	Sunderland	38	11	6	21	36	59	−23	**39**	
16	Bolton Wanderers	38	9	10	19	36	54	−18	**37**	
17	Fulham	38	8	12	18	38	60	−22	**36**	
18	Reading **(R)**	38	10	6	22	41	66	−25	**36**	Relegation to Football League Championship
19	Birmingham City **(R)**	38	8	11	19	46	62	−16	**35**	
20	Derby County **(R)**	38	1	8	29	20	89	−69	**11**	

Figure 11 – League table

Conclusion

The concept of proof is a powerful one, underpinning many fields within mathematics. In conclusion it is clear there isn't a 'best' proof and the decision of which proof to use is case by case. Just like any problem in maths, proofs can be approached in all manner of ways, and whatever method you feel happiest with should be the 'best' method for you.

As mathematicians, we are also teachers for others. Teachers strive to explain concepts as accessibly as possible, and mathematicians share this goal. It is our job to make our proofs as clear and readable as possible, and not intimidate others.

Ultimately, sound proofs uncover universal truths. It is therefore important that we can trust the maths on display, hence proofs are peer-reviewed. Nonetheless, we should always ensure that we are not being misinformed by what we see.

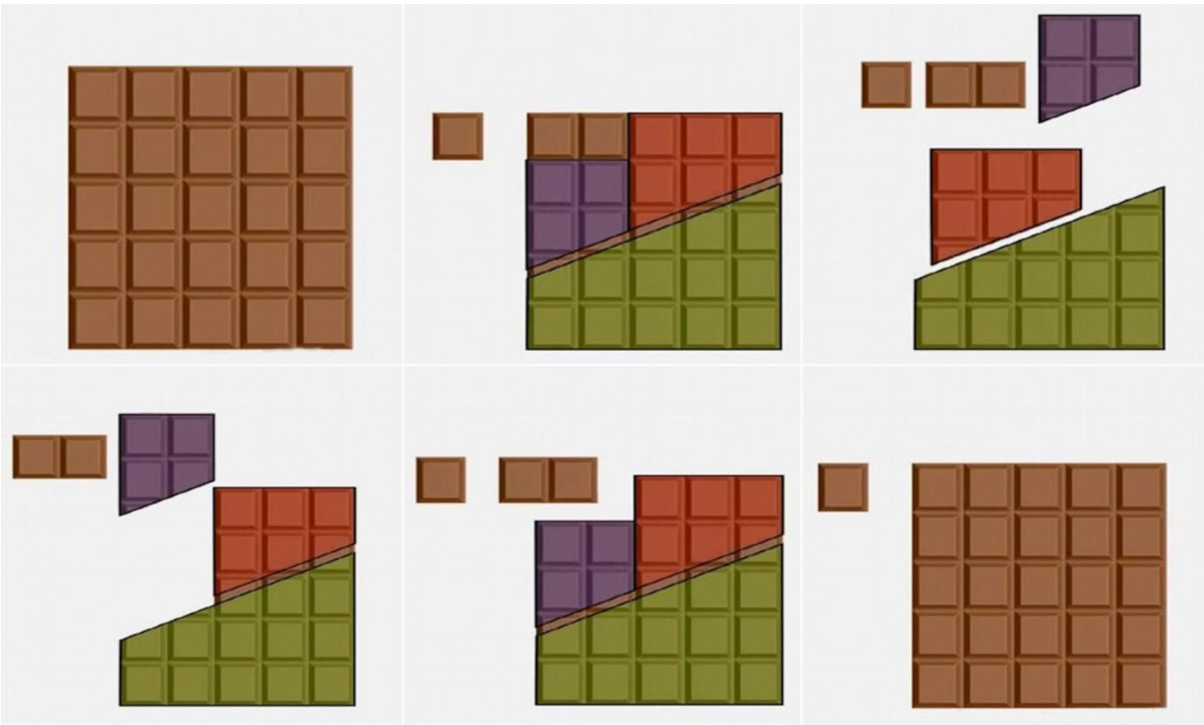

Figure 12 – Misinformation

(1971 words)

Image credits

Figure 1 – Caleb Stanford and https://four-color-theorem.com/2011/03/09/tuttes-map/

Figure 2 – Martin Pelikan

Figure 3 – Made in https://playgameoflife.com – by me

Figure 4 – Taken from published 1966 version

Figure 5 – Rugby World Cup

Figure 6 – Twitter

Figure 7 - Rugby World Cup

Figure 8 – Math Warehouse

Figure 9 - Made in Google Drawings – by me

Figure 10 – James Grime and Wikipedia

Figure 11 – Wikipedia

Figure 12 – Daily Mail

All Latex equations were written by me using https://latexeditor.lagrida.com