The Largest Slice

Arav Ajaykumar

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Introduction

For the 100th anniversary of the World Pizza Society, they decided to make a giant pizza to share among 100 of their closest members. However, they have a strange rule for deciding who gets how much pizza. They order themselves from 1 to 100. Person 1 gets 1% of the whole pizza, Person 2 then gets 2% of what remains, then person 3 gets 3% of what remains after that and so on, ending with the 100th person gets 100% of the remaining pizza. The natural question is, who gets the largest slice of pizza?

Original problem

We define R_n as the amount of pizza remaining after person n takes their share.

Before anyone takes any pizza, all of it remains, so:

$$R_0 = 1$$

Person 1 takes 1% of the pizza and leaves 99%, so:

$$R_1 = R_0 \cdot \frac{99}{100} = 1 \cdot \frac{99}{100} = \frac{99}{100}$$

Person 2 takes 2% of this remaining amount and leaves 98%, so:

$$R_2 = R_1 \cdot \frac{98}{100} = \left(\frac{99}{100}\right)\left(\frac{98}{100}\right)$$

By induction, we can generalise this pattern:

$$R_n = \left(\frac{99}{100}\right)\left(\frac{98}{100}\right)\left(\frac{97}{100}\right)\cdots\left(\frac{100-n}{100}\right)$$

This can be written as:

$$R_n = \frac{99 \cdot 98 \cdot 97 \cdots (100-n)}{100^n}$$

$$R_n = \frac{99!}{100^n(99-n)!}$$

Let the amount of pizza person n gets be P_n.

Person 1 gets 1% of the pizza, so

$$P_1 = \frac{1}{100}R_0.$$

Person 2 gets 2% of what is left after person 1, so

$$P_2 = \frac{2}{100}R_1.$$

It is clear that for person n,

$$P_n = \frac{n}{100}R_{n-1},$$

so

$$P_n = \left(\frac{n}{100}\right)\left(\frac{99!}{100^{\,n-1}(99-(n-1))!}\right).$$

$$P_n = \frac{99!\,n}{100^{\,n}(100-n)!}.$$

We want to find n such that

$$P_{n-1} < P_n > P_{n+1}.$$

First, consider

$$P_{n-1} < P_n.$$

$$\frac{P_{n-1}}{P_n} < 1.$$

Using the formula for P_n,

$$\frac{99!(n-1)}{100^{\,n-1}(100-(n-1))!} \div \frac{99!\,n}{100^{\,n}(100-n)!} < 1.$$

$$\frac{99!(n-1)}{100^{\,n-1}(101-n)!} \cdot \frac{100^{\,n}(100-n)!}{99!\,n} < 1.$$

Like magic, all the large powers and factorials cancel:

$$\frac{n-1}{(101-n)!} \cdot \frac{100(100-n)!}{n} < 1.$$

Since

$$(101-n)! = (101-n)(100-n)!,$$

we obtain

$$\frac{n-1}{(101-n)(100-n)!} \cdot \frac{100(100-n)!}{n} < 1.$$

$$\frac{n-1}{101-n} \cdot \frac{100}{n} < 1.$$

This can be easily turned into a quadratic inequality.

$$100(n-1) \leq (101-n)n.$$

$$100n - 100 \leq 101n - n^2.$$

$$n^2 - n - 100 \leq 0.$$

Using the quadratic formula,

$$n < \frac{1+\sqrt{401}}{2} \approx 10.512.$$

Next, consider

$$P_n > P_{n+1}.$$

$$\frac{P_n}{P_{n+1}} > 1.$$

Substituting the formula for P_n,

$$\frac{99!\,n}{100^{\,n}(100-n)!} \div \frac{99!(n+1)}{100^{\,n+1}(100-(n+1))!} > 1.$$

$$\frac{99!\,n}{100^{\,n}(100-n)!} \cdot \frac{100^{\,n+1}(99-n)!}{99!(n+1)} > 1.$$

All of the factorials and big powers collapse again,

$$\frac{n}{100-n} \cdot \frac{100}{n+1} > 1.$$

This gives

$$100n > (100-n)(n+1).$$

$$100n > 100n + 100 - n^2 - n.$$

$$n^2 + n - 100 > 0.$$

$$n > \frac{-1+\sqrt{401}}{2} \approx 9.512.$$

Overall,

$$\frac{-1+\sqrt{401}}{2} < n < \frac{1+\sqrt{401}}{2}$$

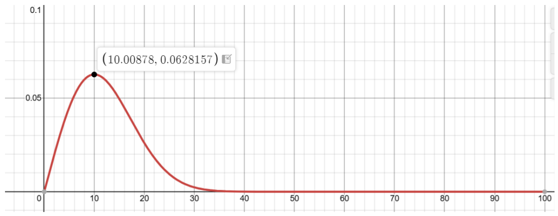

Figure 1: A graph of the sizes of the slice of pizza for each person

The only integer n which holds this inequality is:

$$\boxed{n = 10}$$

Just for fun, the fraction of the whole pizza that person 10 gets would be

$$P_{10} = \frac{99! \cdot 10}{100^{10} \cdot 90!} = \frac{245373636545037}{3906250000000000} \approx 6.28\%$$

No doubt that you could have worked that out on your own without a calculator!

Generalising for any number of people

In the above case, you might have noticed that the percentage of pizza for the largest slice, is very close to $\tau\%$. A natural assumption could be that as the number of people increases, the size of the largest slice tends towards a better approximation of $\tau\%$. Which makes it handy that the World Pizza Society decided that they would like to make another pizza, and share it with "an infinite" number of people.

A reasonable first step would be saying that we have k people, and that person n will get $\frac{n}{k}$ of the remaining pizza.

We can once again use R_n be the fraction of pizza left after the nth person takes their share, with $R_0 = 1$

After person 1 takes their share, there will be $\frac{k-1}{k}$ of the pizza left, so:

$$R_1 = \frac{k-1}{k}$$

Person 2 takes $\frac{2}{k}$ of the remaining pizza, which makes:

$$R_2 = (\frac{k-1}{k})(\frac{k-2}{k})$$

$$P_n = \frac{(k-1)!\, n}{k^{\, n}(k-n)!}.$$

$$R_n = \left(\frac{k-1}{k}\right)\left(\frac{k-2}{k}\right)\left(\frac{k-3}{k}\right)\cdots\left(\frac{k-n}{k}\right)$$

$$R_n = \frac{(k-1)(k-2)(k-3)\cdots(k-n)}{k^n}$$

$$(k-1)(k-2)(k-3)\cdots(k-n) = \frac{(k-1)!}{(k-1-n)!}$$

$$R_n = \frac{(k-1)!}{k^n\,(k-1-n)!}$$

$$P_n = \frac{n}{k} R_{n-1}$$

$$P_n = \frac{n}{k}\left(\frac{(k-1)!}{k^{\,n-1}\,(k-1-(n-1))!}\right)$$

$$P_n = \frac{(k-1)!\,n}{k^n\,(k-n)!}$$

Once again, we want to find n such that

$$P_{n-1} < P_n > P_{n+1}.$$

First, consider

$$P_{n-1} < P_n.$$

This is equivalent to

$$\frac{P_{n-1}}{P_n} < 1.$$

Using the general formula,

$$\frac{(k-1)!(n-1)}{k^{\,n-1}\big(k-(n-1)\big)!} \div \frac{(k-1)!\,n}{k^{\,n}(k-n)!}\} < 1.$$

Rewriting as multiplication,

$$\frac{(k-1)!(n-1)}{k^{\,n-1}(k-n+1)!} \cdot \frac{k^{\,n}(k-n)!}{(k-1)!\,n} < 1.$$

Canceling powers and factorials,

$$\frac{n-1}{k-n+1} \cdot \frac{k}{n} < 1.$$

This gives

$$k(n-1) < n(k-n+1).$$

Expanding both sides,

$$kn - k < kn - n^2 + n.$$

Rearranging,

$$n^2 - n - k < 0.$$

Solving using the quadratic formula,

$$n < \frac{1+\sqrt{4k+1}}{2}.$$

Similarly, for

$$P_n > P_{n+1},$$

we obtain the quadratic

$$n^2 + n - k > 0,$$

which implies

$$n > \frac{-1+\sqrt{4k+1}}{2}.$$

Overall, we have

$$\frac{-1+\sqrt{4k+1}}{2} < n < \frac{1+\sqrt{4k+1}}{2}.$$

As $k \to \infty$, the constants 1 and -1 become negligible, and we see that

$$n \sim \frac{\sqrt{4k}}{2} = \sqrt{k}.$$

A pattern emerges

We can see that for k people, the lucky person who gets the largest slice of pizza tends towards the square root of the number of people. But how big would the slice of pizza be, as $k \to \infty$. Logically, one would expect that the largest slice of pizza would decrease as the number of people increased. At $k = 100$, we saw that the largest slice of pizza was $\frac{0.628}{1}$. Running further tests at $k = 10000, 1000000$ and when $k = 100^a$, and subjecting WolframAlpha to the horrifying calculations of P_n, we get the following results for the size of the largest slice of the pizza

k	Size of the largest slice
100	0.0628156509555294
10 000	0.00608566
1 000 000	0.000606733
100 000 000	6.06551×10^{-5}
10 000 000 000	6.0653×10^{-6}

Table 1: Size of the largest slice as a function of k

As the size of k increases, the size of the largest slice is decreasing as expected, however the bit following the initial zeros seems to be settling on a number, close to 0.60653. Note that this is unfortunately not the similar to the original τ, which is quite the shock given the circular nature of a pizza. If you weirdly like to remember the decimal expansions of random but famous constants, you may have noticed that the actual number, lurking in the depths of this problem, is linked to e, one of the usual suspects to expect for a problem like this. More specifically, the constant here is

$$\frac{1}{\sqrt{e}} = 0.60653...$$

And we can make a guess by looking at it, that the size of the largest slice of pizza

$$\sim \frac{1}{\sqrt{ek}}$$

One could expect e to be hiding somewhere in a problem of "optimizations", and that e also has a close tie to factorials, with

$$\sum_{n=0}^{\infty} \frac{1}{n!} = e$$

And in fact, there is another place where e appears in factorials. This is in Stirling Approximations. The formula is that

$$m! \sim \sqrt{2\pi m} \left(\frac{m}{e}\right)^m,$$

Where the $\sim$ means asymptotically equivalent, as $m \to \infty$, the approximation gets more accurate.

What are Stirling approximations

The Stirling Approximation was initially discovered by de Moivre in 1721, where he stated that

$$n! \sim C\, n^{n+\frac{1}{2}} e^{-n},$$

where C is a constant.

The main bit of the approximation, $(\frac{n}{e})^n$, can be shown like this.

$$n! = 1 \times 2 \times 3 \times \cdots \times n$$

$$\ln(n!) = \ln(1 \times 2 \times 3 \times \cdots \times n)$$

$$\ln(n!) = \sum_{x=1}^{n} \ln(x)$$

When n is large we can use an integral to approximate the sum. Think of a summation "smoothing out" as the number of terms increase:

$$\int_1^n \ln(x)\, dx \approx \ln(n!)$$

This is a fairly common integral with a solution that can be found online, or you can have this as an integral memorised.

$$\int_1^n x\ln(x) - x\,dx = \ln(n!)$$

$$n\ln(n) - n - (\ln(1) - 1) = \ln(n!)$$

$$n\ln(n) - n + 1 = \ln(n!)$$

$$n\ln(n) - n \approx \ln(n!)$$

We can now exponentiate to get the following,

$$e^{\,n\ln(n)-n} \approx n!$$

$$\frac{e^{n\ln(n)}}{e^n} \approx n!$$

$$\frac{\left(e^{\ln(n)}\right)^n}{e^n} \approx n!$$

And here's the final result that we want!

$$\left(\frac{n}{e}\right)^n \approx n!$$

Tedious Limits

Buckle your seatbelts, and get ready to watch a huge fraction cancel out and forming a stunning result We consider

$$P_n = \frac{(k-1)!\,n}{k^n(k-n)!}, \qquad n \sim \sqrt{k}.$$

Using Stirling's formula,

$$m! \sim \sqrt{2\pi}\sqrt{m}\left(\frac{m}{e}\right)^m,$$

we obtain

$$(k-1)! \sim \sqrt{2\pi}\sqrt{k-1}\left(\frac{k-1}{e}\right)^{k-1}, \qquad (k-n)! \sim \sqrt{2\pi}\sqrt{(k-n)}\left(\frac{k-n}{e}\right)^{k-n}.$$

Substituting into P_n,

$$P_n = \frac{\sqrt{2\pi}\,\sqrt{k-1}\,\left(\frac{k-1}{e}\right)^{k-1}\ n}{\sqrt{2\pi}\,\sqrt{k-n}\,\left(\frac{k-n}{e}\right)^{k-n}\ k^n}$$

A monsterous fraction and one which you wouldn't believe works out magically. Fortunately, we can immediately cancel the $\sqrt{2\pi}$ terms:

$$P_n = \left(\frac{\sqrt{k-1}}{\sqrt{k-n}}\right)\left(\frac{\left(\frac{k-1}{e}\right)^{k-1}}{\left(\frac{k-n}{e}\right)^{k-n}}\right)\left(\frac{n}{k^n}\right).$$

We can take the limit of the first fraction first:

$$\lim_{k\to\infty}\sqrt{\frac{k-1}{k-n}}.$$

For any constant a, it is negligible compared to k as $k\to\infty$, if $\frac{a}{k}\to 0$ Since

$$\lim_{k\to\infty}\frac{-1}{k} = 0, \quad \text{we have } k-1\to k,$$

and since

$$\lim_{k\to\infty}\frac{n}{k} = \frac{\sqrt{k}}{k} = \frac{1}{\sqrt{k}} \to 0, \quad \text{we have } k-n\to k.$$

Therefore,

$$\lim_{k\to\infty}\sqrt{\frac{k-1}{k-n}} \to \lim_{k\to\infty}\sqrt{\frac{k}{k}} = \sqrt{1} = 1.$$

We are left with

$$P_n = \frac{\left(\frac{k-1}{e}\right)^{k-1} n}{\left(\frac{k-n}{e}\right)^{k-n} k^n}.$$

Now focus on

$$\frac{\left(\frac{k-1}{e}\right)^{k-1}}{\left(\frac{k-n}{e}\right)^{k-n}}.$$

This is equivalent to

$$\frac{(k-1)^{k-1}e^{-(k-1)}}{(k-n)^{k-n}e^{-(k-n)}} = \frac{(k-1)^{k-1}}{(k-n)^{k-n}}\cdot\frac{e^{-k+1}}{e^{-k+n}}.$$

The right fraction is

$$e^{-k+1-(-k+n)} = e^{1-n}.$$

The left fraction is

$$\frac{(k-1)^{k-1}}{(k-n)^{k-n}} = \frac{\left(k\left(1-\frac{1}{k}\right)\right)^{k-1}}{\left(k\left(1-\frac{n}{k}\right)\right)^{k-n}} = \frac{k^{k-1}}{k^{k-n}}\cdot\frac{\left(1-\frac{1}{k}\right)^{k-1}}{\left(1-\frac{n}{k}\right)^{k-n}}.$$

Hence,

$$= k^{k-1-(k-n)}\cdot\frac{\left(1-\frac{1}{k}\right)^{k-1}}{\left(1-\frac{n}{k}\right)^{k-n}} = k^{n-1}\cdot\frac{\left(1-\frac{1}{k}\right)^{k-1}}{\left(1-\frac{n}{k}\right)^{k-n}}.$$

$$P_n = k^{\,n-1}\left(\frac{\left(1-\frac{1}{k}\right)^{k-1}}{\left(1-\frac{n}{k}\right)^{k-n}}\right)\left(\frac{n}{k}\right)$$

$$P_n = \left(\frac{\left(1-\frac{1}{k}\right)^{k-1}}{\left(1-\frac{n}{k}\right)^{k-n}}\right)\left(\frac{n}{k}\right)$$

The value

$$n = \sqrt{k},$$

so the right fraction is

$$\frac{\sqrt{k}}{k} = \frac{1}{\sqrt{k}}.$$

Unfortunately, I could not figure out how to simplify the righthand fraction, given my current knowledge of maths, which is frustrating; however, a guess will have to be made that it is

$$\frac{1}{\sqrt{e}}.$$

which would give us the final answer.

Conclusion

Wow, that was quite a grind to not even get a result, but given the simple beauty of the expected result, I'm sure there is a way to simplify that last fraction. I still have a few unanswered questions which I could not manage to solve with my current knowledge of maths at the minute, such as, who gets more, the total of all the evens or all the odds. Also, dabbling in probabilities is on my mind, and seeing if you randomly arranged the percentages from 1-100, what is the chance that the first person gets the most pizza. This essay was inspired by the Pythagoras Pie problem, published by the Guardian. Huge thanks to everyone who has taught me maths online and in person. And of course, many thanks to everyone for reading, and for the opportunity to write an essay about maths.