A Optimistically Rational View of an Irrational World

Van Kiet Tam

April 2026

1 Introduction

What is a number?

We are starting off with a strong one, but I want the readers to stop for a moment and ponder this question more in depth. Take a bathroom break right now(even if you don't feel like it), and just come up with some ideas while you are staring listlessly at the bright florescent lighting.

I will be returning to this question again and again throughout this essay, so think deeply about it.

You are done?

Well, the answer might be anti-climatic, but numbers were initially made to represent 'quantities'. Historically, some of the earliest usage of numbers was inscribed on clay tablets for counting land and grain in agriculture. Thus, the first "definition" of numbers that we had was just the set of natural numbers $\mathbb{N}$ containing all positive whole numbers.

I want to bring into light, the exact specifics of a time when we contemplated the 'nature' of numbers.

2 Are Negatives and Irrationals numbers?

2.1 Historical Context

If I were to ask a modern person today whether or not negative numbers can be considered actual "numbers", they would immediately scoff at me and move on. However, if I were to ask an ancient Greek mathematician, they would also immediately scoff at me, but then vehemently oppose my suggestion.

To ancient Greek mathematicians, their primary focus was on geometry. In fact, the three biggest problems of their time were all geometry problems. When viewed through that lens, negative area or negative length makes little physical sense, so the Greeks mostly discarded negative numbers as an idea, and would try to intentionally rephrase proofs and problems to avoid the use of negative numbers.

In a similar vein, rational numbers were accepted since geometry inherently has a lot to do with proportions and ratios such as through Thale's Theorem or the trigonometric functions, but irrational numbers were instead met with some challenges in their adoption because of how they cannot be constructed using just proportions and ratios. In fact, there is a famous myth where the mathematician who discovered $\sqrt{2}$ drowned due to this blasphemy of discovering an irrational number.

However, unlike negative numbers, irrationals were eventually begrudgingly accepted due to there being plausible geometric interpretations for them, and also since most of the push-back came from the Pythagoreans and their philosophy which adapted over time with the wider mathematical community of ancient Greece.

2.2 Exploration: Constructable Numbers

2.2.1 What are Constructible Numbers?

Although this essay is primarily a fun discussion into the nature of numbers, it would feel wrong if I were to not introduce some actual mathematics too.

As mentioned, the ancient Greeks liked to primarily work with geometry. However, their type of geometry is quite different from today. All they had, and all they considered 'valid' to use was a compass to draw circles with a given radius and center, arcs, and copy lengths, together with a straightedge to draw lines, rays, etc. Points can be created when lines intersect, line and arc intersect, outside of just the endpoints of line segments or the centers of circles.

Note that a straightedge does not have tick marks to denote units of length, so you cannot use it to measure. In practice though, you can just define your own unit of length to circumvent this issue. As in, you define some line AB to have length 1 for example, and proceed like normal.

Either way, the ancient Greeks were obsessed with what could be 'constructed' using pure geometry. One of their areas of study was lengths.

Impressively, they were able to construct geometric equivalents to addition, subtraction, multiplication, division, and the square root operation.

To illustrate, draw a line (a) with length a, line (b) with length b, and a ray FG(such that $(a), (b), FG$ are all distinct).

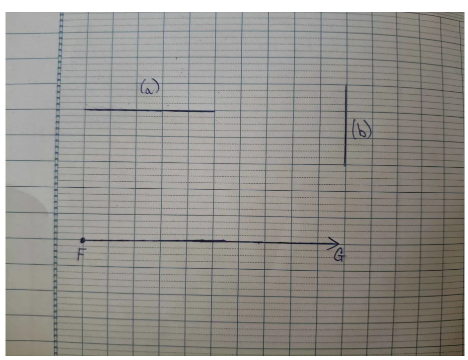
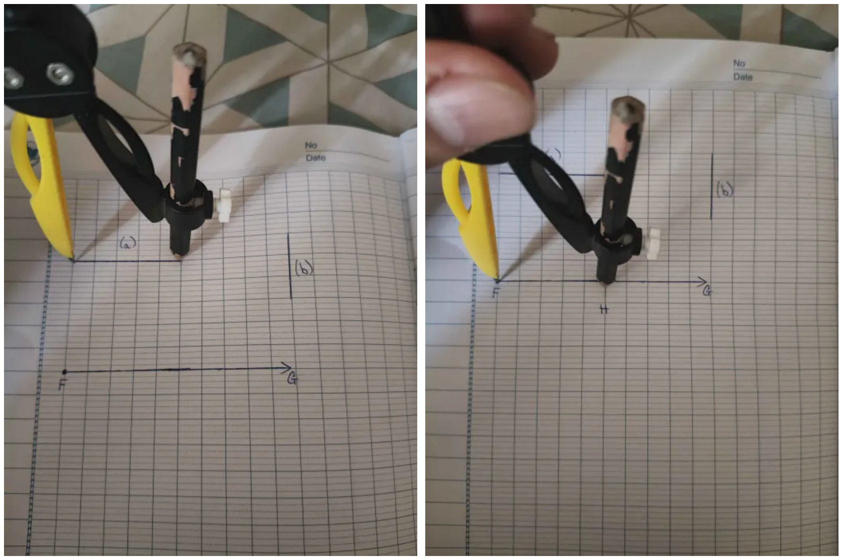


Figure 1: The set-up

Use the compass to transfer the length of line (a) onto the ray FG, drawing an arc centered at F and radius a intersecting FG at H.

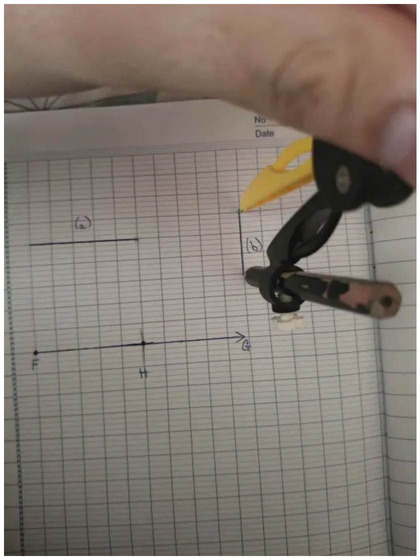


(a) Copying length a

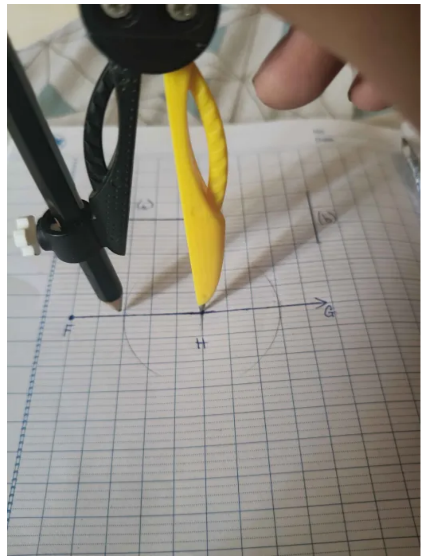
(b) Construct the point H

Figure 2

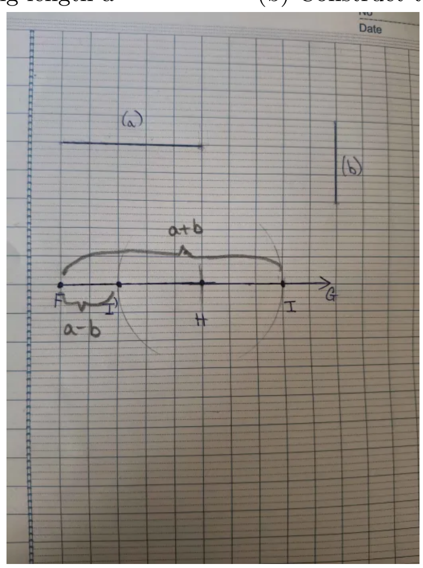
Do the same for line (b). However, draw the arc centered at H instead, with radius b. Depending on whether you draw the point I' or I pictured in the photo below, you get $a - b$, or $a + b$.



(a) Copying length a



(b) Construct the point I or I'



(c) Labelling the diagram

Figure 3

I won't elaborate in depth how you multiply or divide, but you can try using Thales's Theorem as mentioned above, alongside the fact that you can draw parallel lines in geometric constructions.

For square roots, to find the square root of length p, you inscribe a right-triangle inside a circle like in this picture:

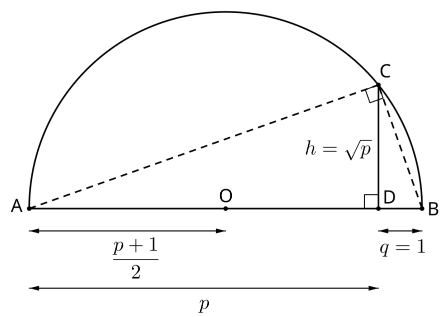


Figure 4: Construction of square root

By triangle similarity, you have the following:

$$\frac{h}{1} = \frac{p}{h}$$
$$h^2 = p$$
$$h = \sqrt{p}$$

Note that since the compass can be used to copy lengths, you can apply any of the above procedures in any way you like. For example, $\sqrt{\sqrt{2}-1}$ just needs you to construct $\sqrt{2}$ first, then copy the length to use to construct $\sqrt{2}-1$, and finally use your trusty compass to measure the length again to construct $\sqrt{\sqrt{2}-1}$.

It is also at this point that I must come clean to the fact that unfortunately I was not the genius who came up with these constructions. Most of the premises and postulates of geometric constructions was instead outlined in Euclid's "Elements". The rules of using straightedge and compass actually comes from the first three postulates, and Euclid was able to prove many theorems using just these simple postulates.

Some of the theorems or constructions proved in the book, which I have implicitly used, is Thales's Theorem, how you can copy lengths using a compass, and the construction of parallel lines.

The set of all numbers which were able to be constructed by the ancient Greeks are known as **the constructible numbers** . This set of numbers encompasses the rationals, but does not cast as wide a net as the reals. Algebraically, it is the set of all positive numbers you can possibly construct using the 5 operations $+, -, \div, \times, \sqrt{}$. For example, $\sqrt{\frac{3}{7} + \sqrt{2\sqrt{7}-1}}$, 20, or $\frac{1}{2}$ are all constructible numbers.

2.2.2 The Three Mathematical Conundrums of Ancient Greece

You might have remembered the fact that I mentioned that the three most important problems at the time in ancient Greece were all geometry problems. Now that you understand the basics of geometric constructions, I can elaborate on what those problems were:

1) Doubling the cube: If given the edge of a cube, can you construct the edge of a second cube whose volume is doubled the first cube?

2) Squaring the circle: If given a circle, can you construct a square with equal area?

3) Trisecting an Angle: If given an arbitrary angle, can you construct an angle equal to one third of it?

The ancient Greeks viewed these problems to be difficult, but not entirely impossible. Sadly enough, during the entire existence of ancient Greece's mathematical existence, none have managed to solve any of the above questions.

However... Place your bets! Place your bets! Of these three problems, which one do you think we have managed to construct ever since the time of the ancient Greeks?

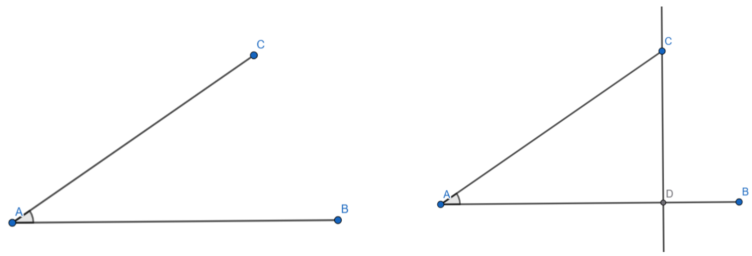
If you guessed "None!", you would be correct! In fact, mathematicians have managed to prove that it is impossible to construct any of the solutions using only straightedge and compass. Can you guess why?

The same line of reasoning actually permeates each one of the problems

1) Doubling the cube: If the first cube has an edge of 1, then its volume is also 1. If you double the volume, then the edge is $\sqrt[3]{2}$.

2) Squaring the circle: If the circle has radius 1 then its area is π. For a square to have the same area, its edge has to be $\sqrt{\pi}$

3) Trisecting an Angle: Note that if an angle θ is constructible, then $\cos(\theta)$ has to be constructible also. There will be a small visual proof below for the case where $0° < \theta < 90°$, but trigonometric identities can be used to extend it to the rest of the cases. The proof of impossibility centers on whether or not some certain $\cos(\theta)$ are constructible, like $\cos(20°)$



(a) The given angle θ

(b) Construct AD, DC. Thus, giving us $\cos(\theta)$

Figure 5: Why θ being constructible means $\cos(\theta)$ is constructible

When it was purely geometric, it seemed reasonable that perhaps some answer existed. However, when phrased algebraically, it causes us to question whether the question is even possible.

For "Doubling the Cube", the fact that we needs to construct a cube root is a red flag considering that we have exclusively dealt in square roots thus far.

If you are "Squaring the Circle", the problem occurs with whether π is constructible in the first place.

"Trisecting an Angle", as I have hinted, does not have a procedure which works for every arbitrary angle, which you can show by picking $20°$ as mentioned and showing its cos is impossible to construct.

In other words, all of the solutions to the three problems are **transcendental numbers** .

The actual full proofs are still quite the involved processes. But I will cut it short here, since this section was just to get a taste of the constructible numbers. If you wish to know more, Another Roof has made a fantastic video on this!

2.3 Why Did These Numbers Feel Wrong?

Generally speaking, if you were to leave a mathematician alone over an extended period of time in his natural habitat, he would begin abstracting definitions. If that period of time was multiple centuries, then he would begin abstracting the notion of "numbers" too until it becomes "thing that can be used to count, measure, and label, and we can hopefully do arithmetic with it".

We take for granted the set of real numbers $\mathbb{R}$ that most students are taught with today. This essay is mainly to illustrate how fluid the definition of numbers can be. Even historical and cultural circumstances can play a part in shaping the intuition we have for what we consider to be numbers.

So, I ask again. Do you consider negatives and irrationals to be numbers? At this point, I hope most of your answers to still be a vehement "yes!". It would be sad if fun maths history trivia can displace the intuition the education system has instilled in you for decade(s) now.

As is evident, I did not write this essay to just demean the ancient Greeks. This is a broader discussion on abstraction versus intuitiveness. The ancient Greeks' intuition surrounding numbers is arguably greater than the average Joe's today. But because of their reliance on a specific intuition, they were unable to answer their questions regarding constructible numbers, which were ultimately more easily solved if one abstracted away the geometric context in the first place.

Abstraction has given us our current system of real numbers $\mathbb{R}$ that is sufficient for any contexts in our daily lives, but the merit of focusing on a specific context is the intuition which it builds. And even a wrong intuition is better than no intuition at all, since pointing out errors in our intuition is what often most leads to finer appreciation for abstraction.