The Beauty of Self-Similar Fractals

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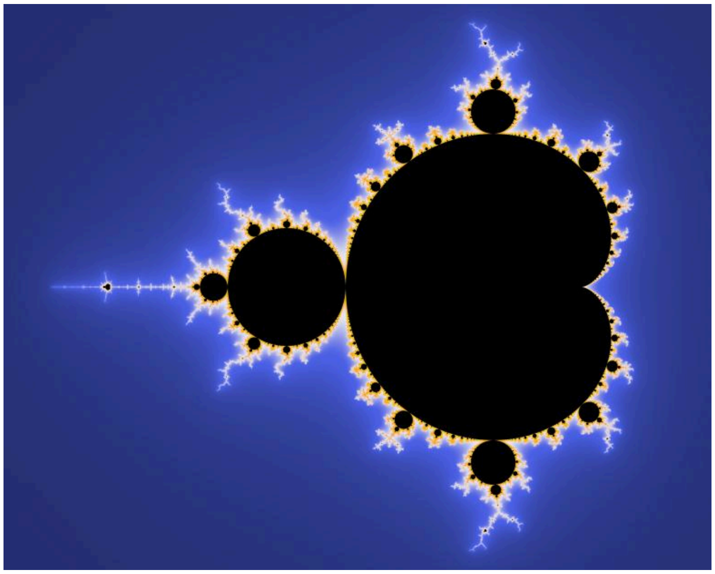
1 Introduction

Fractals are often associated as intricate patterns that emerge from the simplest of shapes, accompanied by wonderful gradients of colour that pronounce their beauty to our eyes.



The Mandelbrot set is a famous example. Image from https://www.stockvault.net/photo/207319/mandelbrot-set

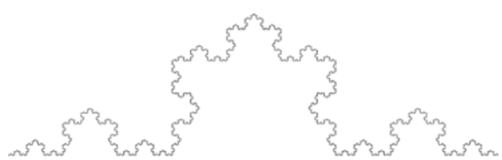
Self-similar fractals or semi self-similar fractals are known for their patterns, whilst **non self-similar** fractals are defined by a deeper property of fractals.

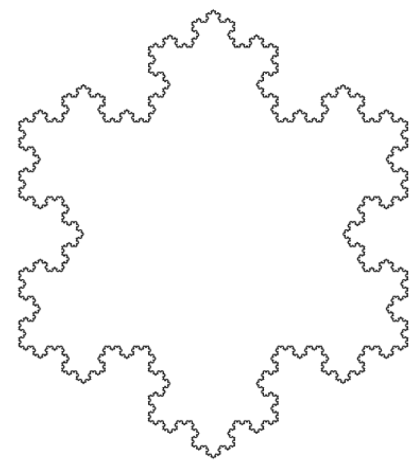
In this paper I will introduce self-similar fractals and hopefully present to you their beauty.

2 The Koch Snowflake

2.1 The Koch Curve

The Koch Curve made by Helge von Koch describes a curve that has no gaps in it and is made of only sharp corners, like a zigzag. It is pronounced with a guttural ‘ch’ as the word ‘koch’ has German origin. If you can’t pronounce it, it is suitable to say ‘coke’.



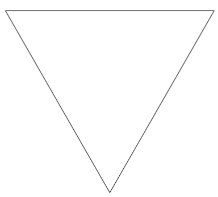


From the Koch Curve, we make the Koch Snowflake by adding two more curves and joining them end to end. Pretty cool right? (Pun intended).

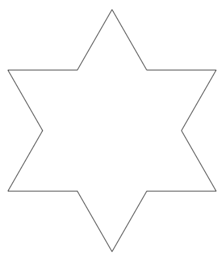
2.2 Construction

Fractals can be made by repeating a single process many times: the Koch Curve and Snowflake were made by repeating the same process to a starting shape. Below, I demonstrate how to construct them.

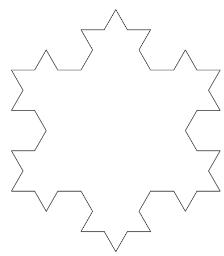
Note: these are not my images. I give image credit to Huo Chen for making the fractal generator found here: https://codinglab.huostravelblog.com/math/fractal-generator/index.php?fractal=KOSN&step=5&size=500&color=000000



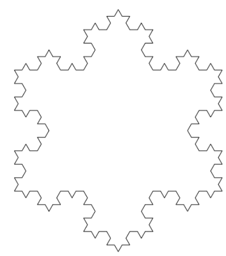
Starting triangle



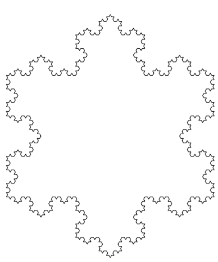
Iteration 1



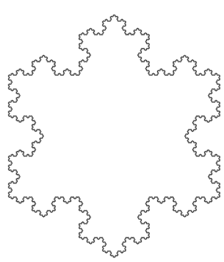
Iteration 2



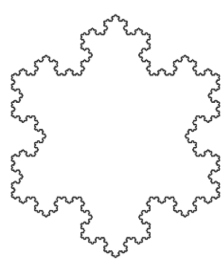
Iteration 3



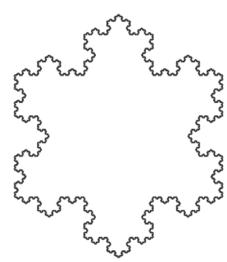
Iteration 4



Iteration 5



Iteration 6



Iteration 7

Steps to making the Koch Snowflake:

1. Start with an equilateral triangle.

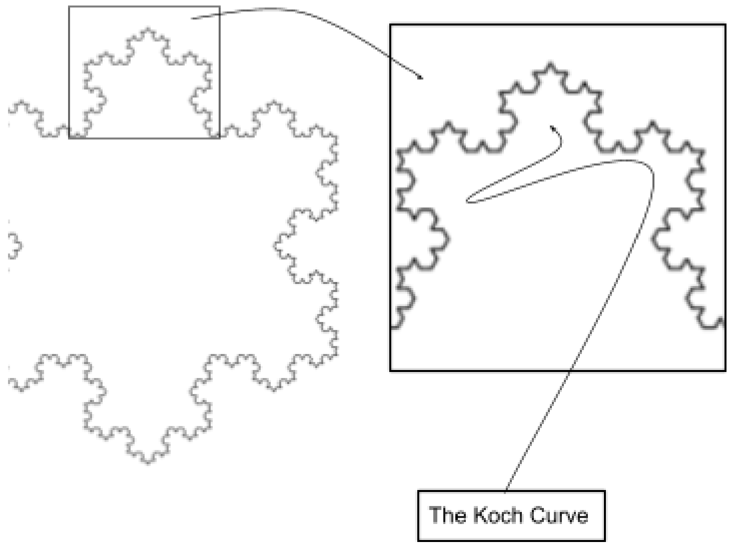
2. “Poke out” the middle of each line to form a triangle.

3. Repeat step 2 until satisfied.

We can repeat step 2 as many times as we want to get higher resolution fractals, although beyond iteration 4 they are hard to discern on this document.

If you zoom into the fractal, you would see that they look similar to the fractal zoomed out. This is due to its self-similar nature.

Note: I made this image using the above images.



2.3 Properties of the Koch Snowflake

I want to ask you **two** questions. First, if we infinitely repeat step 2 to make the Koch Snowflake, what will its **perimeter** be? Second, what will its **area** be?

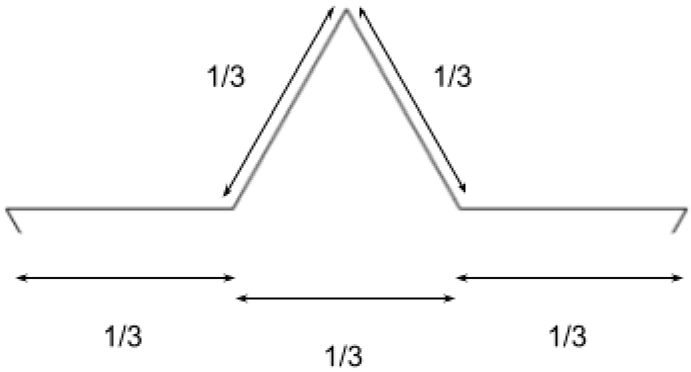
As we increase the repetitions, will it approach some **fixed** value for the perimeter / area, or will it **increase to infinity**?

2.3.1 Perimeter

For the perimeter, we simplify the problem by focussing on the top Koch Curve of the Koch Snowflake. At the end of calculations we can multiply by 3 to get the total perimeter.

Let the top Koch Curve have length **1**. You may find it helpful to follow along the iterations from the images above.

Iteration 1: If we 'poke out'' a triangle, we are dividing the curve into **4 parts of length** $\frac{1}{3}$., so the total length is $\frac{4}{3}$**.**



Iteration 2: We 'poke out' 3 more triangles on each of the previous triangles. Looking at the left $\frac{1}{3}$ segment from the above image, we see that it'll become $\frac{4}{9}$ from what happened previously in Iteration 1. Now we multiply that by 4 since this process happens to 4 sides, and we get the length is $\frac{16}{9}$.

Iteration 3: We 'poke out' 3 more triangles on each of the previous triangles. Looking at the left most $\frac{1}{9}$ segment from the image of Iteration 2, we see that its length will multiply by $\frac{4}{3}$ to get $\frac{4}{27}$. Now there are 16 $\frac{1}{9}$ length sides, so we get $16 \times \frac{4}{27} = \frac{64}{27}$

So far, we've got the lengths in what seems like a pattern:

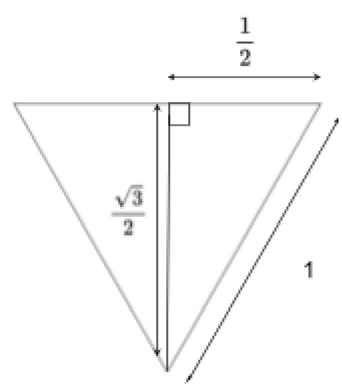
Iteration 1: $\frac{4}{3}$, Iteration 2: $\frac{16}{9}$, Iteration 3: $\frac{64}{27}$

If you know about geometric sequences, then you'll recognise that the common ratio r is $\frac{4}{3}$. Since $r > 1$, the sequence will diverge to infinity. If you think about multiplying something by a value greater than 1, it will increase, no matter how much it is greater than 1. For example, $4 * 1.01 = 4.04, 4.04 > 4$.

What does this mean? This means that the Koch Snowflake has an **infinite perimeter**. Intunitely, this is because as we zoom in to the snowflake there is more detail, and measuring it will result in a longer length, which increases without bound.

2.3.2 Area

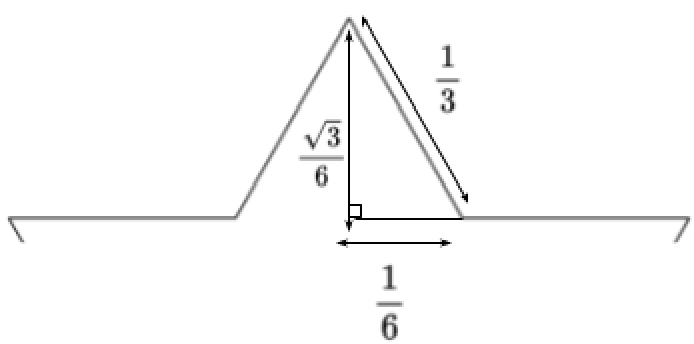
Let the sides of the Koch Snowflake have length **1**. For the area, we focus on the whole shape this time, and go through our step wise process of finding the area.
Original triangle: We divide the equilateral triangle in half to find its height.



Now we use **Pythagoras's theorem** $A^2 + B^2 = C^2$ where A and B are the sides that are **not the hypotenuse**, so in our case, the **height of the triangle** and $\frac{1}{2}$ **respectively**. Since we know the hypotenuse length (1) but need to know the height, we can **rearrange** to get $A^2 = C^2 - B^2$ and now **substitute** in values to get $A^2 = 1 - (\frac{1}{2})^2 = 1 - \frac{1}{4} = \frac{3}{4}$ **Square rooting** our answer will give us out height, and we get $\frac{\sqrt{3}}{2}$.
Now we can use our area formula for a triangle (base x height divided by 2) to get $(1 \times \frac{\sqrt{3}}{2}) \div 2 = \frac{\sqrt{3}}{4}$ as the area for the original triangle.

Iteration 1: We calculate this by adding the area of the original triangle to 3 of the area of the smaller triangle formed. Doing the previous process to the different lengths gives us an area of $\frac{\sqrt{3}}{36}$.
Here's proof:
$A^2 = C^2 - B^2$
$A^2 = (\frac{1}{3})^2 - (\frac{1}{6})^2 = \frac{1}{9} - \frac{1}{36} = \frac{3}{36}$
$so,\ A = \frac{\sqrt{3}}{6}$, so area of the smaller triangle $= \frac{1}{3} \times \frac{\sqrt{3}}{6} \div 2 = \frac{\sqrt{3}}{36}$

So the area of our three smaller triangles is $\frac{\sqrt{3}}{36} \times 3 = \frac{\sqrt{3}}{12}$.



Iteration 2: We now find the area of the even smaller triangle through the same method (but I won't go through it for brevity) to achieve an area of $\frac{\sqrt{3}}{324}$ for each triangle, and since there are 12, the total area of the smallest triangles is $\frac{\sqrt{3}}{27}$.

Iteration 3: We now find the total area of the tiniest triangle, and it comes out to be $\frac{4\sqrt{3}}{243}$

We can express the **total** area of the triangle up until iteration 3 as $\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{12} + \frac{\sqrt{3}}{27} + \frac{4\sqrt{3}}{243} = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{12}(1 + \frac{4}{9} + \frac{16}{81}) = \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{12}(1 + \frac{4}{9} + (\frac{4}{9})^2)$

Hmm… this looks like **a pattern**. It appears that the total area of the Koch Snowflake will continue like $\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{12}(1 + \frac{4}{9} + (\frac{4}{9})^2 + (\frac{4}{9})^3 + (\frac{4}{9})^4 + ...)$

To save words and your time, the pattern does continue, and so we have $\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{12}(a_n)$ where a_n is the geometric sequence with starting term 1 and common ratio $\frac{4}{9}$. Using the formula for the **infinite sum** of a geometric sequence we can work out the infinite sum, and then the area of the Koch Snowflake.

$S_\infty = \frac{1}{1-\frac{4}{9}} = \frac{1}{\frac{5}{9}} = \frac{9}{5}$ so the total area of the triangle is $\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{12}(\frac{9}{5}) = \frac{2\sqrt{3}}{5}$

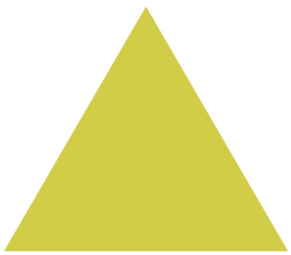
So it turns out that the area of the Koch Snowflake is **finite**, whilst its perimeter is **infinite.**

This isn't just for the Koch Snowflake original side lengths 1, we can replace the area formula with $A + \frac{A}{3}(\frac{9}{5}) = \frac{8}{5}A$**.**

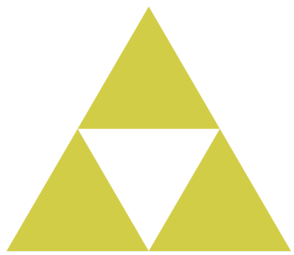
3 Sierpinski Triangle

3.1 Construction

Image credit is the same as for the Koch Snowflake iterations.

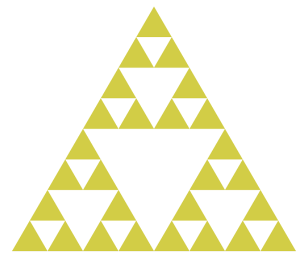


Starting triangle

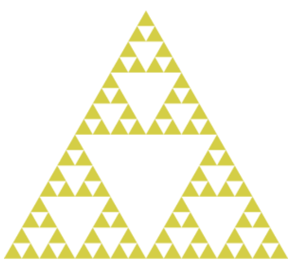


Iteration 1

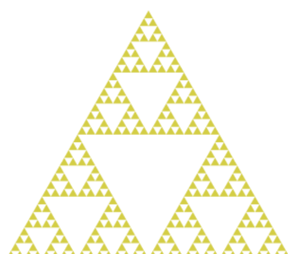
Iteration 2



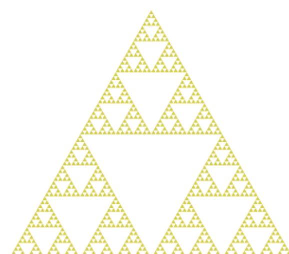
Iteration 3



Iteration 4



Iteration 5



Iteration 6

Iteration 10

Steps to making the Sierpinski Triangle:

1. Start with an equilateral triangle.

2. In the middle of each triangle, cut out an upside down triangle

3. Repeat step 2 until satisfied.

3.2 Properties of the Sierpinski Triangle

We see that as the iterations progress, the triangle becomes more faint, with iteration 10 being borderline invisible.
Let us make a guess for the perimeter and area of the Sierpinski Triangle using this knowledge and our previous experience of calculating the same for the Koch Snowflake.

3.2.1 Perimeter

Let the sides of the original triangle be of length 1. Therefore, it has perimeter 3.
Iteration 1: Each side is divided in half, leaving 3 triangles of perimeter $\frac{3}{2}$, giving the total a perimeter of $\frac{9}{2}$
Iteration 2: Each side is divided in half again, giving us 9 triangles of perimeter $\frac{3}{4}$ and a total perimeter of $\frac{27}{4}$
Iteration 3: Each side is divided once more, giving us 27 triangles of perimeter $\frac{3}{8}$ and a total perimeter of $\frac{81}{8}$
We can clearly see this is a geometric sequence with common ratio $\frac{3}{2}$, and since $\frac{3}{2} > 1$, the perimeter will increase to **infinity** as we keep iterating.

3.2.2 Area

Let the sides of the original triangle be of length **1**. Therefore, it has area $\frac{\sqrt{3}}{4}$.
We lose $\frac{1}{4}$ of the area with each iteration, so iteration **1** will have an area $\frac{3\sqrt{3}}{16}$.
Iteration 2 will have an area $\frac{9\sqrt{3}}{64}$, iteration 3 will have an area of $\frac{27\sqrt{3}}{256}$, and iteration **n** will have an area of $\frac{\sqrt{3}}{4} \times (\frac{3}{4})^n$. We can see that the area gets smaller and smaller visually, but to mathematically show this we can use a limit:

$$\lim_{n\to\infty} (\frac{3}{4})^n = 0$$

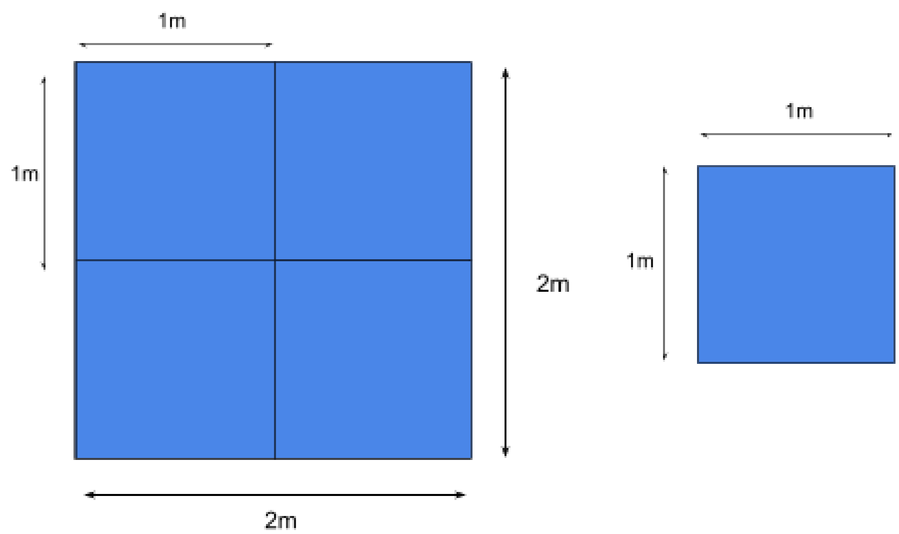
This should make sense, because any number multiplied by a number smaller than 1 will decrease, and get closer to 0. We can also see that no matter the initial starting area, the area of the triangle will approach 0, as multiplying any number by 0 gives 0.
For example, $4 \ * \ 0.99 = 3.96, \ 3.96 < 4.$

So, the Sierpinski Triangle also has a **finite** area yet **infinite** perimeter, but in this case the **area is 0.**

4 Fractal Dimensions

When you think of the word ‘dimension’, you probably think of 1D lines, 2D shapes, and 3D objects. 1D objects only have length, 2D objects have area, and 3D objects have volume.

I want you to think of dimensions in another way. Let’s say we have a square and cut it into 4 pieces.



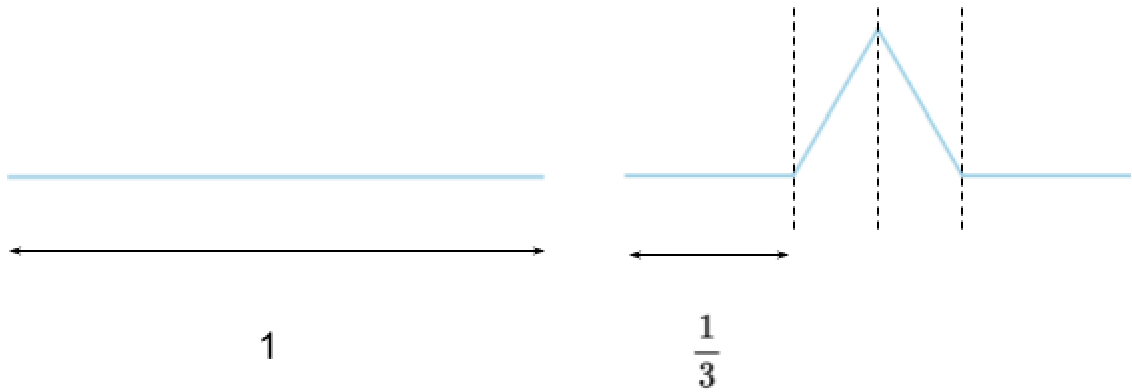
Its side lengths are multiplied by a scale factor of $\frac{1}{2}$, and the number of pieces we’ve split it into is 4.
In 3D, if we have a cube of side lengths 2, then split it into 8 equal pieces, then each piece will have side length 1, so the scale factor is $\frac{1}{2}$ and the number of pieces we’ve split it into is 8.

For 2D, we had a scale factor of $\frac{1}{2}$ and 4 pieces. For 3D, we had a scale factor of $\frac{1}{2}$ and 8 pieces. Can you see a pattern?

It looks like the reciprocal of the **scale factor (r)** raised to the **dimension (D)** will give us the **number of pieces (N).** We have $(\frac{1}{2})^2 = 4,\ (\frac{1}{2})^3 = 8$. More generally, we can write a formula for the dimension of any object to get:
$r^D = N$

By logarithm laws, we can calculate that $D = log_r N$.
Using our new definition, we can find fractal dimensions.

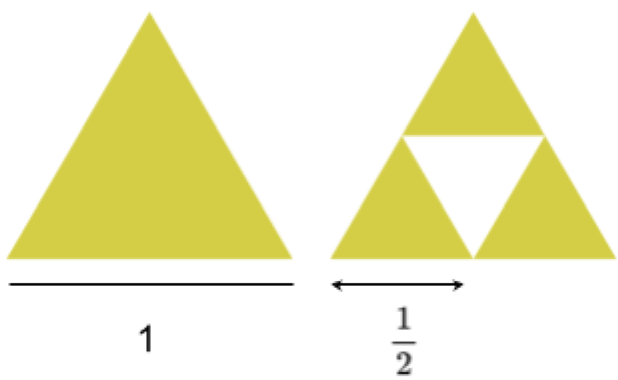
4.1 Koch Curve



We divide the length of the original line by 3, so our scaling factor is $\frac{1}{3}$, so $r = 3$ and $N = 4$ since the number of objects turns from 1 to 4, and $D = log_3 4 = \sim 1.262$

So the dimension of the Koch Curve is approximately 1.262

4.2 Sierpinski Triangle



We halve the side lengths of the original triangle, so our scaling factor is $\frac{1}{2}$, so $r = 2$, $N = 3$, since the number of objects turns from 1 to 3, and $D = log_2 3 = \sim 1.585$

So the dimension of the Sierpinski Triangle is approximately 1.585.

4.3 Coastlines and applications

Coastlines are actually fractals as well, just not the self-similar ones we've covered so far. Now that we know about fractal dimensions, we can state a better definition for a fractal:

Definition: A fractal is a shape with a fractal dimension.

If we measure the coastline of Britain using smaller and smaller rulers, our measurement for the coast length will increase forever because the coastline is rough and has fractal properties, like the Koch Snowflake and Sierpinski Triangle.

I will not go over how to calculate the fractal dimension in detail, but one method is we put the country on a grid, count the amount of squares the coastline covers, then increase the size of the country / use smaller squares and count again. This gives us data to analyse the fractal dimension or roughness of the coastline.

To learn more this article may be helpful:
https://en.wikipedia.org/wiki/Coastline_paradox#Mathematical_aspects

Fractal dimensions represent roughness, so from 1D to 2D, a fractal close to 1D will be very smooth, and a fractal close to 2D will be very rough. This has applications in quantifying roughness as this linked paper explores.
https://www.sciencedirect.com/science/article/abs/pii/S001379440200019X#preview-section-abstract

5 Conclusion

In conclusion, fractals are wonderful pieces of art that, despite having contradicting properties, seem to work just fine. They also have fascinating real-life applications in quantifying roughness and measuring our coastlines.