How physics and maths beat the stock market.

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The Black –Scholes – Merton Model

Its Creation

The Black- Scholes- Merton Model (BSM Model) was a model created by financial theorists: Fischer Black, Robert Merton, and Myron Scholes, in 1973. The first equation created was published in the "Pricing of Options and Corporate Liabilities" in the Journal of Political Economy.

As a result of the creation of this model, it landed Merton and Scholes as both the winners of the 1997 Nobel Memorial Prize in Economic Sciences. Unfortunately, Black had died in 1995 making him ineligible for the Nobel prize.

Purpose

Options are the contracts which enables the investor in the future to take a specific action needed to benefit you in the stock market.

Call options are contracts that gives the shareholders the right to acquire a share of a company's stock in a later period for a fixed price (strike price). This advantage the shareholder if price begins to increase in the future but can also disadvantage them as they would pay for the option and the option price would be worthless after expiry date. Furthermore, the European call option (which the model is based off) only allows the contract to take effect on the expiry date and no time before then.

Put options are contracts that give the shareholders the right to sell a share of company's stock later at the strike price. Therefore, the shareholder could prevent a potential loss from occurring if the price plummets. Yet again, the European call option forces the contract to occur at the date of expiration and never before or after and therefore makes the option costly and ineffective in the long-term. (Veritasium ,2024)

Shareholders are not obliged to use these options if price has reverse predictions.

Hedging is a risk management strategy where investing is required to reduce potential losses, like insurance. By holding onto the opposite position in a buyer seller for a related good, the investor can reduce losses for if one investment drops and a hedge gains value, the loss can be offset.

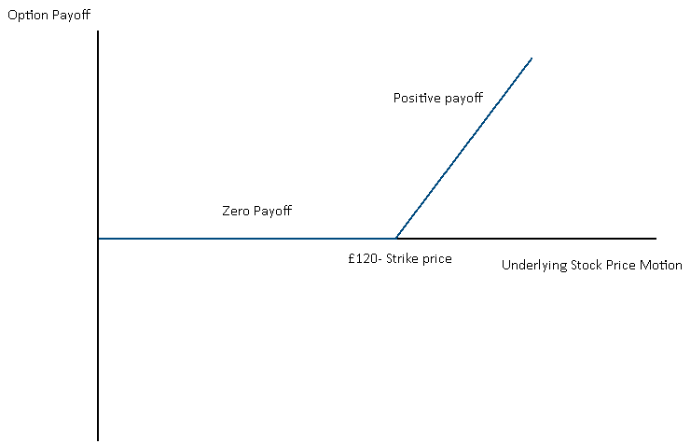

Figure 1- A graphical representation of how Option prices and Stock prices relate in a linear motion

The Actual Model

The Black-Scholes-Merton Model can be broken down into 3 significant models:

-The partial differntial equation

-The European call option price model

-The European put option price model

The partial differential equation for Black Scholes equation is as follows:

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

A partial differntial equation has a multivariable function meaning there are multiple augments for a function like f (X, Y, Z). The partial derivative is for multiple variables; its derivative is in one of those variables while the other variables are held constant.

This PDE is a linear second derivative of the parabolic type; linear PDEs means the unknown variable are linear like time; second derivative PDE means a PDE up to the second derivative. In this case, the second derivative is gamma. (Ivrii, V.,n.d.)

Stochastic equations:

Models on how a variable would evolve over time and therefore we can derive the probability for a certain event to occur in that time.

The Brownian Motion (Wiener Process)

The four main properties that follow BM is that it starts at 0 and it is a continuous function in time.

Further assumptions to be made is one that the event in the future is independent of the event in the past

The Brownian Motion models random behaviour over time in the fields of physics and finance. In the case of physics, it was used to model the randomness of motion of particles in fluids.

Louis Bachelier was a French mathematical student when in 1990, he published his PhD thesis where he borrowed the idea of Brownian Motion and implemented into mathematical financial where it can be used to predict stock prices in the Paris Stock Exchange.

Furthermore, his inspiration of ball motion in a Galton board gave way to how he viewed the motion of stock prices- like a normal distribution centred on the current price spreading out over time.

A Galton Board is a mathematical physical device to demonstrate the Gaussian (Normal) distribution. Using small balls dropped at the top and colliding with many pegs in an array to create a pyramid. This device tries to portray probability and Normal distribution for the number of events by the different pegs a ball has collided with to reach the bottom.

In finance, it was used for the fluctuations of asset prices in the market. The assumption for stock prices to implement Brownian Motion was that there was constant change over a small period.

Geometric Brownian Motion

A standard Brownian Motion takes on negative values, which do not make sense in the stock market as it would mean negative stock prices, so the Black- Scholes Model assumed a new motion called the GBM or Geometric Brownian Motion. The probability distribution used in Black Scholes model was log- normal making it different from the one used in Bachelier Model and therefore not producing negative asset prices.

$$dS = \mu S dt + \sigma S dW$$

This equation determines stock prices with two components:

-drift which is the growth rate

-Volatility represents the uncertainty

Taylor's Series

The label on approximation of non-polynomials with polynomials when near to x=0.

This provides a decent explanation for the small angle approximation $cos\theta = 1 - \frac{\theta^2}{2}$ due to approximating $cos\theta$ with a quadratic gives this formula. (3Blue1Brown ,2017)

Ito's Lemma and its derivation

Ito's Lemma can be derived heuristically using the Taylor's series expansion for a function f(X,t) where x/X=S in the PDE equation. X is the stochastic process. (University of Exeter,n.d.)

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}dX + \frac{1}{2}\ \frac{\partial^2 f}{\partial x^2}(dX)^2_{+}\ldots$$

Now since x/X= S, dX= dS so substitute the geometric Brownian motion for dX.

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial x}(\mu Sdt + \sigma SdW) + \frac{1}{2}\ \frac{\partial^2 f}{\partial x^2}(\mu Sdt + \sigma SdW)^2$$

Looking at the (dX)^2 term, we can invoke a multiplication rule where:

-dt^2=0

-$dt \times dW$ =0

-dW^2=dt

Therefore,

$$(dX)^2 = (\mu Sdt + \sigma SdW)^2 = S^2\mu^2dt^2 + 2S^2\mu\sigma dtdW + S^2\sigma^2dW^2$$

$$= S^2\mu^2(0)^2 + 2S^2\mu\sigma(0) + S^2\sigma^2(dt)$$

$$= S^2\sigma^2dt$$

$$df = \frac{\partial f}{\partial t}dt + \mu\frac{\partial f}{\partial x}Sdt + \frac{\partial f}{\partial x}\sigma SdW + \frac{1}{2}\ \frac{\partial^2 f}{\partial x^2}(S^2\sigma^2dt)$$

$$df = \left(\frac{\partial f}{\partial t} + \mu\frac{\partial f}{\partial x}S + \frac{1}{2}\sigma^2S^2\frac{\partial^2 f}{\partial x^2}\right)dt + \sigma S\frac{\partial f}{\partial x}dW$$

Furthermore, since f=V and (x/X) =S, substitute the two variables into the equation to output the final Ito's Lemma.

$$dV = \frac{\partial V}{\partial t}dt + \frac{\partial V}{\partial S}dS + \frac{1}{2}\frac{\partial^2 V}{\partial S^2}dS^2$$

$$dV = \left(\frac{\partial V}{\partial t} + \mu S\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2}\right)dt + \sigma S\frac{\partial V}{\partial S}dW$$

Derivation of the Black- Scholes Model PDE

The first thing to create is a risk-free portfolio which follows dynamic hedging. Dynamic hedging is the frequent adjustments to the hedging position to align with the portfolio and keep up with the implied volatility. (Chen, J. ,2019)

Assume V represents the option contract, where the parameter s is the current stock price and parameter t is current time.

Create an equation for a risk- free portfolio of hedging where V is the long-position option and short-position Δ S is an unknown number of stocks sold against the option.

$$\Pi = V - \Delta S$$

Find the 1st derivative of the equation.

$$d\Pi = dV - \frac{\partial V}{\partial S} \Delta dS$$

Substitute Ito's lemma and geometric Brownian function into $d\Pi$

$$d\Pi = \left(\frac{\partial V}{\partial t} + \mu S \frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - \mu S \frac{\partial V}{\partial S}\right) dt + \left(\sigma S \frac{\partial V}{\partial S} - \sigma S \frac{\partial V}{\partial S}\right) dW$$

$$d\Pi = \left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2}\right) dt.$$

Now- we can equate the return of a portfolio Pi with the value of a portfolio growing at a risk-free rate.

$$d\Pi = r\Pi dt,$$

$$\left(\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \ \frac{\partial^2 V}{\partial S^2}\right) dt = r\left(V - S \ \frac{\partial V}{\partial 2S}\right) dt$$

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

$$0 \leq S, 0 \leq t \leq T$$

Now breaking down each derivative, $\frac{\partial V}{\partial S}$ is known as delta or the rate of change in option over change in stock price. $\frac{\partial^2 V}{\partial S^2}$ is known as the rate of change of delta over the chance in stock price. Furthermore, $\frac{\partial V}{\partial t}$is known as the rate of change in option price over change in time.

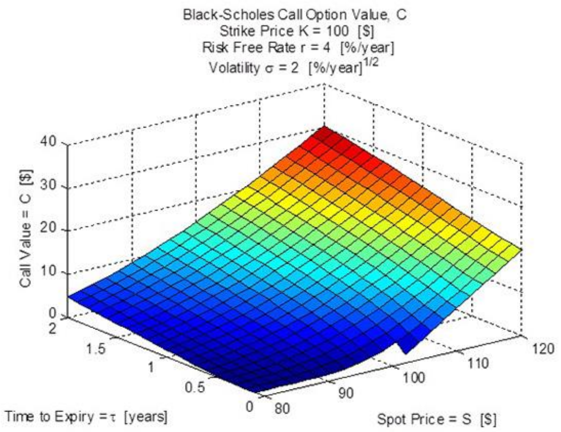

Figure 2. A 3-D graph of how t, S and C relate for a multivariable function. (Certainerror.com,2026)

Solving the PDE for the European Call Option

Now, we solve it. The solution is derived from a different version of the PDE. When the PDE was first created, it was an equation of movement. Another equation of movement was the heat equation developed by Joseph Fourier in 1789. Just like the BSM model captured how option value V had changed with time t as well as the underlying stock price S change tied to the option. In the heat option, the equation related temperature T changes as both time and depth changes. Both equations are similar with the fact that they both contain a first order derivative in respect to time and up to the second derivative in respect to another variable (space for Heat eq and S for BLM model). (Joyner, Charles D ,2016)

So, by converting the BSM equation into the heat equation, the solution can be found.

Heat equation for reference:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

The time to maturity is (T-t) where T is the time of maturity and t is the current time.

Establishing boundary conditions for S=0 and S getting closer to infinity.

Call option is likely to be exercise d when S closes in on infinity to receive the largest benefit possible from the option.

V in this case in the call option C.

$$V(S,T) = f(S) = \max(S - E, 0)$$

$$V(0,t) = 0$$

Assign appropriate values to S and t.

To explore further on why these values are taken, I recommend the Georgia Southern University's take on explaining the BSM model.

$$S = e^x$$

$$t = T - \frac{\tau}{\sigma^2}$$

$$V(S,t) = v(x,\tau) = v\left(\ln(S), \frac{\sigma^2}{2}(T-t)\right)$$

Derive the respective derivatives to align with the derivatives from the BSM PDE.

$$\frac{\partial V}{\partial t} = \frac{\partial v}{\partial \tau}\frac{\partial \tau}{\partial t} = \frac{-\sigma^2}{2}\frac{\partial v}{\partial \tau}$$

$$\frac{\partial V}{\partial s} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial S} = \frac{1}{S}\frac{\partial v}{\partial x}$$

$$\frac{\partial^2 V}{\partial S^2} = \frac{\partial}{\partial xs}\left(\frac{\partial v}{\partial S}\right) = \frac{\partial}{\partial S}\left(\frac{1}{S}\frac{\partial v}{\partial x}\right)$$

$$= \frac{1}{-S^2}\frac{\partial v}{\partial x} + \frac{1}{S^2}\frac{\partial^2 v}{\partial x^2}$$

Now input new values into the PDE equation.

$$\left(-\frac{\sigma^2}{2}\frac{\partial v}{\partial \tau}\right) + \frac{1}{2}\sigma^2 S^2\left(\frac{1}{-S^2}\frac{\partial v}{\partial x} + \frac{1}{S^2}\frac{\partial^2 v}{\partial x^2}\right) + rS\left(\frac{1}{S}\frac{\partial v}{\partial x}\right) - rV = 0$$

$$\left(-\frac{\sigma^2}{2}\frac{\partial v}{\partial \tau}\right) - \frac{\sigma^2}{2}\frac{\partial v}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 v}{\partial x^2} + r\frac{\partial v}{\partial x} - rV = 0$$

Let us simplify the equation so it becomes easier to carry throughout the essay.

$$-\frac{\sigma^2}{2}v_\tau - \frac{\sigma^2}{2}v_x + \frac{\sigma^2}{2}v_{xx} + rv_x - rV = 0$$

$$-\frac{\sigma^2}{2}v_\tau + \frac{\sigma^2}{2}v_{xx} + \left(r - \frac{\sigma^2}{2}\right)v_x - rV = 0$$

Multiply by $-\frac{2}{\sigma^2}$

$$v_\tau - v_{xx} + \left(\frac{\sigma^2 - 2r}{\sigma^2}\right)v_x + \frac{2r}{\sigma^2}v = 0$$

$$v_\tau = v_{xx} + \left(\frac{2r}{\sigma^2} - 1\right)v_x - \frac{2r}{\sigma^2}v$$

Now since we multiplied by a non-numerical value, let us assign K for $\frac{2r}{\sigma^2}$ and τ for t since K is a constant and easier to arrange [Remember r is constant]

$$v_t = v_{xx} + (k-1)v_x - kv$$

$$v(x,t) = e^{\alpha x + \beta t}(x,t) = \phi u$$

Where, the exponent is φ and the (x,t) is u

This substitution eliminates the two terms on the far right.

$$\frac{\partial v}{\partial t} = \beta\phi u + \phi \frac{\partial u}{\partial t}$$

$$\frac{\partial v}{\partial x} = \alpha\phi u + \phi \frac{\partial u}{\partial x}$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial}{\partial x}\left(\alpha\phi u + \phi \frac{\partial^2 u}{\partial x^2}\right)$$

$$= \alpha^2 \phi u + 2\alpha\phi \frac{\partial u}{\partial x} + \phi \frac{\partial^2 u}{\partial x^2}$$

Now, substitute these derivatives into the latest version of the PDE with k.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + [2\alpha + (k-1)]\frac{\partial u}{\partial x} + [\alpha^2 + (k-1)\alpha - k - \beta]u$$

A positive of this equation is the exponent term is common so is cancelled out.

The two arbitrary constant presents in the equation: alpha and beta, can take any value to accustom for the PDE transformation. In addition, the condition must be altered to suit these added terms and PDE transformation.

Therefore, assign the two constants:

$$\alpha = -\frac{k-1}{2}$$

$$\beta = \alpha^2 + (k-1)\alpha - k = -\frac{(k+1)^2}{4}$$

Now the two coefficients for the far-right terms are 0 so a 1-D heat equation forms.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

The new bounds for variable x and t

$$-\infty < x < \infty$$

$$0 \leq t \leq \frac{\sigma^2}{2}T$$

Now, with the heat equation, we can use the well-known heat equation solution with an integral to reverse all our transformations to give the analytical solution of the PDE.

$$u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} u_0(s) e^{-\frac{(x-s)^2}{4\tau}} ds$$

This would be the analytical solution for the Black Scholes Model for European call option. The put option would reverse this model so $P(S,t) = N(d_2)Ke^{-r(T-t)} - N(d_1)S$

$$C(S,t) = N(d_1)S - N(d_2)Ke^{-r(T-t)}$$

$$N = \int_{-\infty}^{y} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$$

$$d_1 = \frac{\ln\left(\frac{S}{E}\right) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

When C is the call option price,

- N: cumulative distribution function of the normal distribution
- S: stock price of an asset
- K: strike price (price determined by option)
- r: risk-free interest rate
- T-t: time to maturity
- Sigma: volatility of asset

This model turns a stochastic process into a deterministic model, but a few assumptions must be made.

1) Lognormal distribution- Assumes stock prices follow lognormal distribution as asset prices cannot go beyond 0 (is not negative)
2) Expiration date- Assumes option is exercised only at date of expiration representing the European call option
3) Random walk- Assumes volatility follows the random walk as market direction (up/down) can never be predicted and is truly random
4) Risk-free interest rate- Assumes interest rates are held at constant which can only happen for a short amount of time
5) Normal distribution- Assume return on stock investment are normally distributed as volatility is constant.

Limitations which follow these assumptions:

1) Limited to European Market- Cannot be applied to American market as European market only allows options to be exercised on its expiration date
2) Risk-free interest rates- Assumes constant interest rates but never a reality, especially in the economic term

3) Constant volatility – Implied volatility is never constant as assumed with Random Walk, especially under market stress
4) Frictionless market- Assumes no transaction costs but usually not ever true in reality

Conclusion

The Black Scholes Model is most definitely the most popular and one of the most important mathematical financial models of our times. It has led to the rise of trillion dollar worth branches like securitised debts and has reduced market inefficiency. But in the discipline of maths, this equation proves the worth of mathematicians to come on top of the stock market.

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