The Bubble That Knew Too Much

From the Steiner Problem to the Double Bubble Theorem

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Introduction

Imagine you have been given the task of connecting four cities, each at a corner of a square, via roads, but you have to minimise the costs, so you need to make the shortest possible network. What are you going to do? Are the diagonals the best paths? Or is it some other network? This is the Steiner problem — and it turns out, the solution is in bubbles!

That's right, you can solve this problem at home. Here's what you do: make a soapy water solution by mixing liquid dish soap with water, then take a wire and bend it into a cube. Dip the wire frame into the water, and look at it from one side to see the two-dimensional solution.

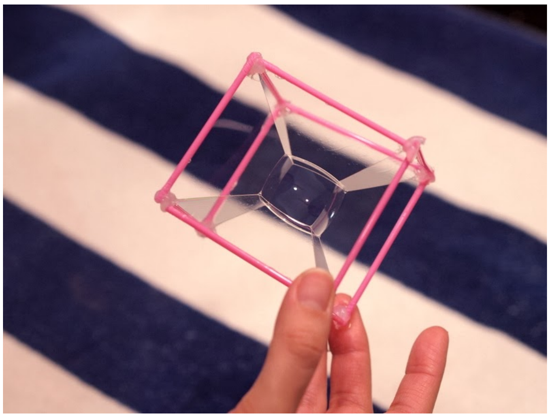
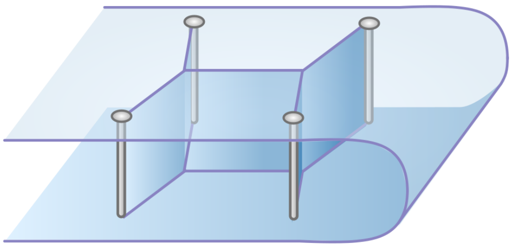


Figure 1: A soap film inside a cubic wire frame

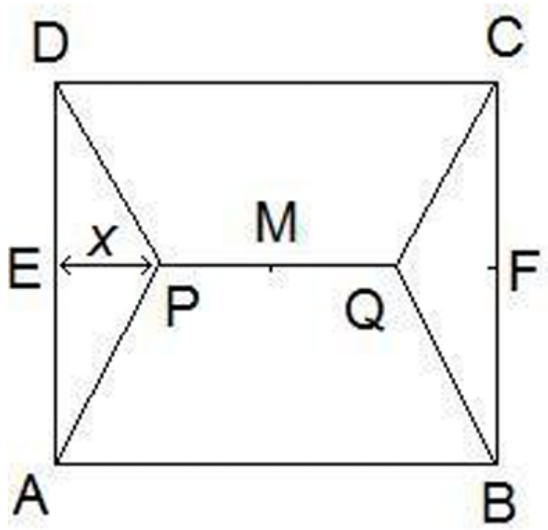
This is what we see. Not an X, but instead two Y-shaped junctions connected upside down. Let $ABCD$ be a unit square (check figure 2). With simple trigonometry, we can show that the total distance is $1 + \sqrt{3} = 2.732$, which is less than the sum of the diagonals $= 2\sqrt{2} = 2.828$.

This isn't a party trick — this is nature. Nature always finds a way to make an optimal solution that we humans struggle to prove.

Differentiation and the Steiner Problem



(a) The Steiner problem solution in a soap bubble



(b) The two-dimensional Steiner problem diagram

Figure 2: Two views of the Steiner problem

We can prove our previous result rigorously using differentiation. Let x be the distance $EP = QF$. Let $L(x)$ be the total network length. Then:

$$L(x) = 4\sqrt{0.5^2 + x^2} + 1 - 2x$$

Using the chain rule with $u = 0.5^2 + x^2$:

$$\frac{d}{dx}L(x) = \frac{2}{\sqrt{u}}u' - 2 = \frac{4x}{\sqrt{0.5^2 + x^2}} - 2$$

To minimise the length, we find the stationary point by setting the derivative to zero:

$$L'(x) = \frac{4x}{\sqrt{0.5^2 + x^2}} - 2 = 0$$

$$2x = \sqrt{0.5^2 + x^2} \implies 4x^2 = 0.5^2 + x^2 \implies 3x^2 = 0.5^2 \implies x = \frac{1}{2\sqrt{3}}$$

So our minimum length is:

$$L\left(\frac{1}{2\sqrt{3}}\right) = 1 + \sqrt{3}$$

We can confirm this trigonometrically. Setting:

$$\angle DPA = \angle DPQ = \angle APQ = \angle CQP = \angle CQB = \angle BQP = 120^\circ$$

we get:

$$x = 0.5\tan 30^\circ = \frac{1}{2\sqrt{3}} \implies y = 1 + \sqrt{3}$$

The soap film didn't just guess the answer — it solved a calculus problem and got it right, instantly, every time.

The important question is: why 120°? It all lies in a little bit of physics. At the point where three bubble films meet, surface tension causes each film to exert an equal force pulling outward, making the bubble taut. Writing the force vectors along each film, and noting that the system is in equilibrium:

$$\vec{F_1} + \vec{F_2} + \vec{F_3} = \vec{0}$$

By symmetry, let $\vec{F_1}$ and $\vec{F_3}$ make equal angles θ with $\vec{F_2}$:

$$F_1 \sin\theta = F_3 \sin(180^\circ - \theta)$$

$$F_1 \cos\theta + F_3 \cos(180^\circ - \theta) = F_2$$

This gives $F_1 = F_2 = F_3$, and therefore $\theta = 60^\circ$. The angle between any two forces is 120° — the same angle we obtained by minimising $L(x)$. Minimising length and balancing forces give the same answer.

We have covered the two-dimensional proof, but what about three dimensions? What if there are not just four points but infinitely many — that is, an entire surface? How do we minimise the surface area for a fixed volume? The answer is a sphere. Bubbles are round and not cubic because surface tension minimises the surface area for a fixed volume, and the sphere is the unique shape that achieves this.

Have you ever noticed that when you blow bubbles, they always stick together in specific structures? When two bubbles touch, they form two rounded domes with a film between them. Why? I always used to wonder how this worked, until I read about Plateau's Laws and the Double Bubble Theorem.

Plateau's Laws and the Road to the Double Bubble

One of the most stubborn conjectures to be proved, the Double Bubble Theorem remained unproven until the year 2000. The theorem states that the standard double bubble is the most efficient shape for enclosing two given volumes with minimal surface area. To understand it, we must first go back to the 19th century. Belgian physicist Joseph Plateau studied soap films experimentally and described how they behave. His two most important laws are:

1. The mean curvature of a soap film is constant at every point on the same piece of film.
2. Soap films always meet in threes along an edge called a *Plateau border*, at an angle of $\arccos\left(-\frac{1}{2}\right) = 120^\circ$.

That's right — we are back to 120°, and this number appears everywhere in nature.

The hexagons in a beehive have interior angles of 120°. Bees are solving the same optimisation problem: minimising total wall length for a fixed area. The only shapes that tile a flat surface without gaps are equilateral triangles, squares, and regular hexagons, and of these, the hexagon is the most efficient. While this was first conjectured in the 4th century, it was not formally proved until 1999 by mathematician Thomas Hales. It took almost 2000 years to prove what bees had been doing correctly all along.

A year later, mathematicians turned to a harder problem. Proved in the year 2000, the Double Bubble Theorem states that the standard double bubble — two rounded caps with a curved dividing wall, all meeting at 120° — is the most efficient possible structure for enclosing two given volumes.

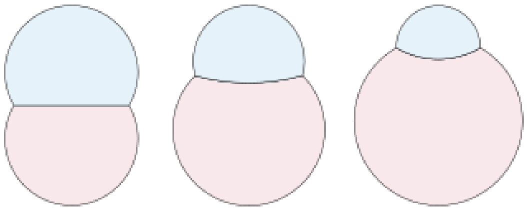


Figure 3: The standard double bubble shape

What made the theorem so hard to prove is that there are infinitely many possible surfaces, and checking each one is beyond impossible. Mathematicians needed a single argument that would eliminate all the alternatives at once — something elegant and structural.

The study of soap films began with Plateau in the 19th century. Plateau's laws on soap film structure were formalised by Jean Taylor in 1976. In 1991, Joel Foisy was the first to precisely state the three-dimensional Double Bubble Conjecture in his undergraduate thesis. In 1995, Hass and Schlafly proved the equal-volume case — but only by reducing it to 200,000 numerical checks carried out by a computer in 20 minutes. This sparked genuine debate: is a proof that no human can verify by hand still valid mathematics? The question became moot in 2000, when Hutchings, Morgan, Ritoré and Ros proved the full conjecture without a single line of computer code, instead showing that every non-standard double bubble is unstable — it can always be perturbed to reduce surface area.

Mean Curvature

Before we understand the proof, we need to understand mean curvature. We can define curvature as how fast a curve bends. For a circle of radius r, the curvature is simply $\frac{1}{r}$ — a smaller circle bends faster. In three dimensions, at any point on a surface we can find two principal curvatures by slicing through the surface with two perpendicular planes. The mean curvature is their average:

$$H = \frac{1}{r_1} + \frac{1}{r_2}$$

For a single sphere with radius r, both principal curvatures are equal, so $H = \frac{1}{r}$ — constant everywhere, exactly as Plateau's first law requires.

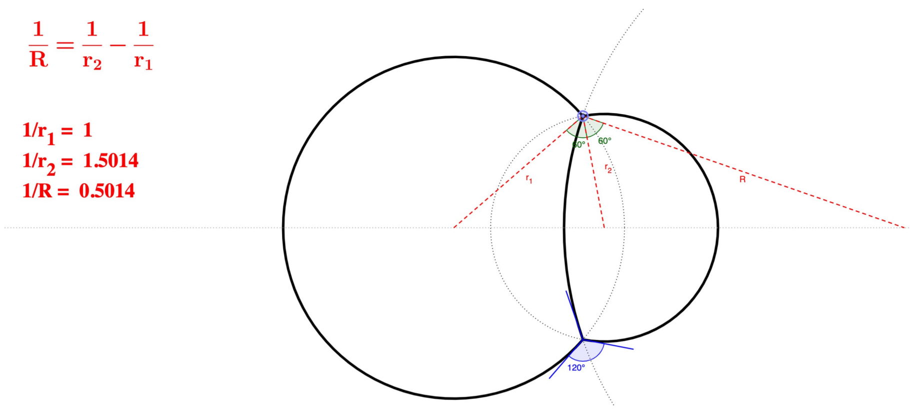


Figure 4: A double bubble with radii R_1, R_2, and R_3 labelled

For a double bubble, $H_1 = \dfrac{1}{R_1}$ is the mean curvature of the larger cap and $H_2 = \dfrac{1}{R_2}$ is that of the smaller cap. We define H_3 as:

$$H_3 = H_2 - H_1 = \frac{1}{R_3}$$

where R_3 is the radius of the dividing wall. If $H_3 = 0$, the wall is flat, meaning the two bubbles have equal curvature. Otherwise, the wall curves toward the larger bubble. You can verify this by blowing some bubbles — and have fun doing it.

The Young-Laplace Equation

Now we come to the final and most important part: the Young-Laplace equation. Consider a bubble of radius R that expands by a small amount dR. We know $W = F \times d$ and $F = PA$, so the work done by the pressure difference ΔP is:

$$W = \Delta P \cdot 4\pi R^2 \, dR$$

A soap bubble has two surfaces — inner and outer — giving a total surface area of $8\pi R^2$. The surface energy is $E = T \times 8\pi R^2$, where T is surface tension, a constant property of the soap solution. The work done equals the increase in surface energy:

$$\frac{dE}{dR} = 16\pi T R \implies dE = 16\pi T R \, dR$$

Setting $W = dE$:

$$\Delta P \cdot 4\pi R^2 \, dR = 16\pi T R \, dR$$

$$\boxed{\Delta P = \frac{4T}{R}}$$

This is the Young-Laplace equation. Since R is in the denominator, a smaller bubble has a greater pressure difference — meaning higher internal pressure. Applying this to the double bubble with radii R_1 (larger) and R_2 (smaller):

$$P_1 = P_0 + \frac{4T}{R_1}, \qquad P_2 = P_0 + \frac{4T}{R_2}$$

Since $R_2 < R_1$, we have $P_2 > P_1$. The pressure difference across the dividing wall is:

$$\Delta P_{\text{wall}} = P_2 - P_1 = 4T\left(\frac{1}{R_2} - \frac{1}{R_1}\right) = \frac{4T}{R_3}$$

which confirms our earlier result $H_3 = H_2 - H_1 = \dfrac{1}{R_3}$. This equation tells us exactly why the wall curves: the smaller bubble is at higher pressure and pushes against the dividing wall. When $R_1 = R_2$, the pressure difference is zero and the wall is perfectly flat — exactly what we observe.

Finally, since T is a property of the soap solution rather than the bubble itself, every film at the junction carries the same surface tension. Each film therefore pulls at the junction with equal force, and as we proved from the Steiner problem, equal forces meeting at a point must be equally spaced — at 120°. The same angle, the same reason, confirmed now in three dimensions through pressure balance. The 120° angle is not a coincidence; it is the inevitable consequence of equal surface tensions.

Applications and Conclusion

How exactly does proving this theorem matter beyond mathematics? These findings appear everywhere, from biology to architecture. The 120° rule is found inside our own cells: biological tissues pack together at 120° to minimise surface area, reducing the energy cost of maintaining cell walls and increasing efficiency. In engineering, Plateau's laws underpin the design of foam-like lightweight structures with exceptional strength. This geometry was even used in the design of the stunning Beijing Water Cube, the aquatics venue for the 2008 Olympics.

The next time you blow bubbles, you are not just watching bubbles form — you are watching nature use calculus, Plateau's laws, and the Young-Laplace equation simultaneously to solve an optimisation problem. We needed differential calculus, vector algebra, and four mathematicians working until the year 2000 to verify what the soap bubble has been doing for millions of years. Nature never needed proof. We did.

Yet there are still things we cannot prove — among them, the triple bubble conjecture, which states that the standard triple bubble is the most efficient shape for enclosing three given volumes.

I always wondered why bubbles seemed to clump together in the same ways, or why it was so hard to stack them differently. While the problem seemed trivial at first, the deeper I dug, the more I realised how difficult it is to prove what nature does effortlessly. We have only scratched the surface. And it still surprises me that, after so many centuries of mathematical progress, there are things that continue to happen all around us every day that we simply cannot prove — yet. That's what makes nature so special.

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