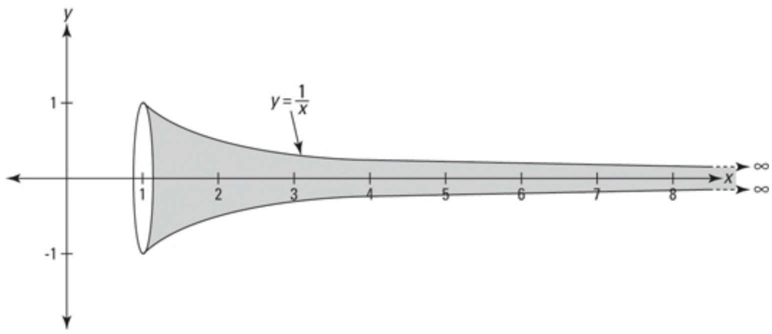
The conflict between Mathematical Rigidity and Human Intuition :

Gabriel’s Horn Paradox

A paradox, formally defined as something that is a seemingly absurd or contradictory statement or proposition which when investigated may prove to be well founded or true, is an example of something that in almost every case proves that there was an error. However, in maths provides evidence that the framework underpinning maths itself is far more conceptually intricate than human intuition could ever be.

At a very basic level, maths is binary: right or wrong. Despite this, as it becomes more advanced and assumptions become more and more embedded within the working, such as infinity or limits, maths is less about intuition, but more to do with intrinsic consistency, ensuring rules and laws are applied in the same manner, even when they might not feel 100% correct intuitively. This thus provides enough evidence to suggest that within maths, paradoxes are not proof that maths breaks down, but more that the assumptions that we make aren’t always intuitively sound. My personal favourite example of a mathematical paradox is the paradox of ‘Gabriel’s Horn’.



Gabriels Horn

Gabriel's horn is formed by rotating the graph of y=1/x about the x axis in a 360° rotation, with the domain being limited to x≥1, forming a 3D shape which is infinitely ‘long’ in the x-direction, as the graph of y=1/x gradually approaches y=0 but never quite reaches it.

The paradox is centred around both the volume and the surface area of this shape, with the surface area being said to be infinite, while the volume is said to be finite. Now, conceptually, I think that the surface area conjecture is fairly self explanatory: if you have an infinitely ‘long’ shape, the area should be infinite. Although this seems fairly self explanatory, there is a mathematical way to prove that the surface area is infinite, because as we have stated so far we know that intuition alone is not sufficient evidence that something is mathematically true.

To prove that the surface area is infinite, we can use the surface area of revolution formula:

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Since we know that y=$\frac{1}{x}$ we can substitute that in for y, and $-\ (\frac{1}{x^2})$ for $\frac{dy}{dx}$. This gives us:

$$S \ = \ \int_1^\infty \frac{2\pi}{x}\sqrt{1 \ + \ (-\ \frac{1}{x^2})^2}\, dx \ = \ 2\pi\int_1^\infty (\frac{1}{x})\sqrt{1 \ + \ \frac{1}{x^4}}\, dx.$$

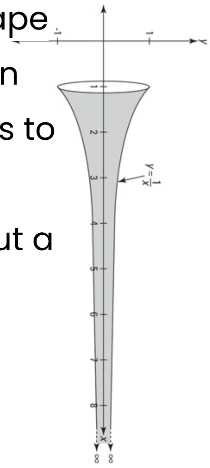
Looking at the equation, we can notice that $\sqrt{1 \ + \ \frac{1}{x^4}}$ will always be greater than 1, and thus that $(\frac{1}{x})\sqrt{1 \ + \ \frac{1}{x^4}} \geq (\frac{1}{x})$, which provides evidence that the surface area will always be greater than $2\pi\int_1^\infty (\frac{1}{x})dx$.

Hence, we can then use standard integration, showing that the surface area of the shape is infinite.

When thinking about this problem for the first time, my brain was immediately more attracted to the idea of the volume of the shape being finite . Although the mathematical proof below is very straight forward, the fact that an endless shape had a finite volume didn't sit right with me, so I tried to approach the problem in different ways, not trying to prove it in another way, but rather try to prove to myself that it was possible.

$$V \ = \ \int_1^\infty \pi(\frac{1}{x})^2 dx \ = \ \pi[-\ \frac{1}{x}]_1^\infty = \pi[1 \ - \ \frac{1}{\infty}] \ = \ \pi.$$

The Tension between Calculation and Intuition



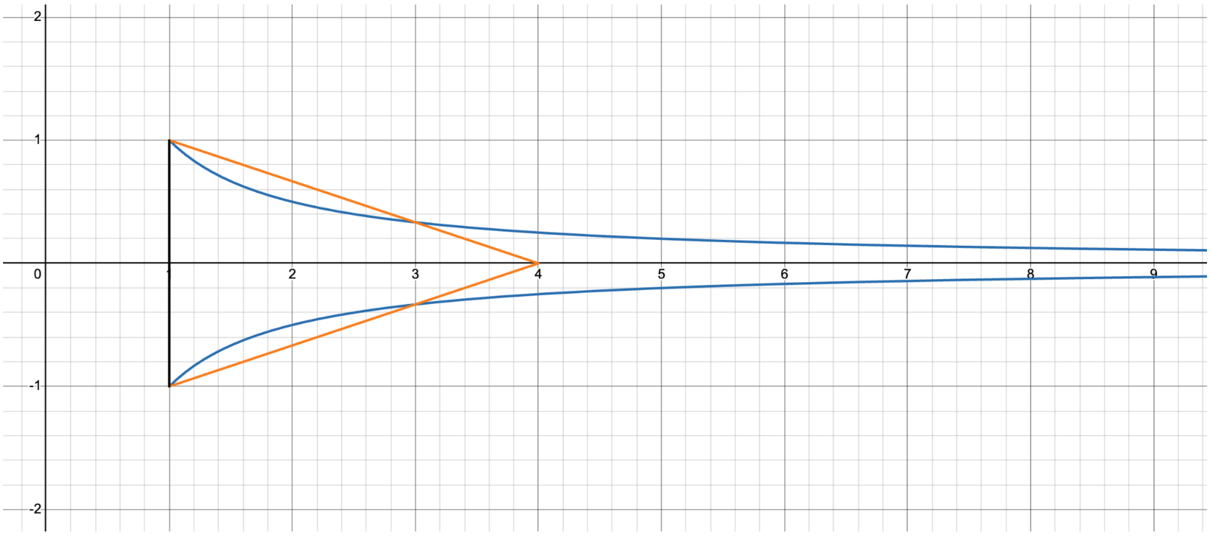
The first way that I approached the problem was thinking about the shape as a tangible object, something that is of course impossible to achieve in reality, however helpful just to visualise the problem. My first thought was to tip the horn onto its side as shown to form a long, bottomless cup. I then began to think about why a liquid which was poured into a vessel without a bottom would ever stop flowing, coming to the conclusion that the diameter of the vessel must reach a point which is smaller than the diameter of the molecule of the liquid which is being poured into it, preventing the liquid from flowing further.

Thinking back to the original graph of y=$\frac{1}{x}$, as the value of x $\rightarrow \infty$, y $\rightarrow 0$. Due to this, if the 'wall' of the vessel, represented by the graph, has any thickness at all, then at some point the walls will close on each other and meet, disobeying the problem as a whole.

These two thoughts about the conditions in which this paradox could persist allowed me to come to these two constrictions:

1. The molecules of whatever liquid is poured in must be infinitesimally small.
2. The 'walls' of the vessel must be infinitesimally thin.

At this point in my thinking, I found it more difficult than ever to conceptually visualise this problem with any element of realism whatsoever, due to the fact that there would be nothing that would prevent the liquid from flowing, consequently leading me to conclude, for a long time, that I simply did not agree with the conjecture that the volume of Gabriel's horn was finite. On top of this, with the surface area being infinite, I became even more hesitant to accept this fact about the volume: If the external surface area is infinite, while the 'thickness' of the wall is effectively 0, then the internal surface area must also be infinite. Assuming my earlier constriction: the molecules of the liquid are infinitesimally small, this must mean that for the vessel to be full, a molecule must be in contact with the entirety of the internal surface area, thus requiring an infinite number of molecules = an infinite volume.



I then began thinking, if the volume of the shape is pi, then there must be a cone with an equal volume which could help me to compare the volumes side by side, hopefully allowing me to start to see how the conjecture could be true. To help me try to visualise the comparison between a cone of volume pi and Gabriel's horn, I used Desmos to provide me with a cross-sectional diagram for a cone of radius 1 and height 3 (orange), alongside Gabriel's horn. Using basic calculation, the volume of the cone comes out to be = pi, and thus supposedly = the volume of Gabriel's horn.

I used this diagram to compare the cross sectional areas of these two shapes, as if the cross-sectional area of the cone is less than the horn, then surely the volume of the shape would be? The section of the graph which is inside the cone but outside the horn was, from what I thought, obviously smaller than the area which extends after the tip of the cone, however if I have learned one thing from researching this paradox, it is that you can not assume things are true based on the way that they are observed.

Hence, I tried to first find out both of the areas to a good degree of accuracy, before then comparing them and seeing the result. During this, I used the fact that the integral of y = 1/x from 1 to infinity is infinite to derive that the area from 4 to infinity is infinite. This hence means that this area exceeds the area outside the horn while inside the cone, which I worked out to be 11/6 - ln(3).

This provided me with what I thought was undeniable evidence that the volume of the cone was not just greater than pi which was what I originally set out to explore, but was also infinite, falling back onto the same intuitive answer, as if the area of the graph was infinite, surely when you revolve it 360° around the x axis, surely you come to an infinite volume. However, I then applied the logic to find the volume of the shape, realising that the volume of the horn does not use y=1/x, it uses y = 1/x^2, due to the fact that the radius is equal to 1/x, which must then be squared and multiplied by pi to find the area. The integral from 1/x^2 converges to one, meaning that it is then very easy to see why the total volume is equal to pi.

The mathematical proof of the finite volume explains this, meaning I became more and more inclined to believing this conjecture:

$$V = \int_{1}^{\infty} \pi(\frac{1}{x})^2 dx = \pi[-\frac{1}{x}]_1^{\infty} = \pi[1 - \frac{1}{\infty}] = \pi.$$

Despite this, I still found myself with underlying doubts, that even something is mathematically provable, does that mean that it is the truth, or does that mean that it merely conforms to the rules and constrictions within maths. This question turns a relatively small inquisition about a single paradox into questioning the entirety of maths, which I would say I have neither the credibility nor the ability to disprove, and for that reason I will stay within the boundaries set out by all the mathematicians before me.

Diving into Convergence

I started to think about converging sequences, as I felt that this was a relatively similar problem if the volumes were split into thin frustums, as the volumes of the frustums would get progressively smaller and smaller until they tend to 0, allowing the cumulative volume to tend to pi, a finite value. This led me to question the nature of convergence, to think about which I thought of my favourite converging sequence:

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots \to 2$$

For a sequence to converge like this, the common ratio between two consecutive terms must be < 1. This is because this means that the number being added will become progressively smaller and smaller, thus meaning that the sum of these terms will progressively get closer and closer without exceeding a specific number, which in this case is 2. With this in mind, I tried applying convergence in the context of Gabriel's horn, splitting it into very vague frustums, which gave me a value of the volume approximately equal to pi, despite the fact that I only went up to x=4, revealing that the value would most likely converge to pi very slowly after that point.

This theory of convergence is what allows the volume to converge to pi, as the integral of 1/x^2 from 1 to infinity converges to 1. This is an example of 'fast decay', as the value converges, whereas the example stated earlier of 1/x is an example of slow decay, meaning that there is a divergence, which is the key difference. Intuitively both graphs are decaying to zero; one converges and the other diverges to infinity. This seems absurd. Intuitively a convergence to a very low finite number and a divergence to infinity seem direct opposites, and how can these opposite results derive from such similar graphs? The answer perhaps is in our understanding of infinity. In fact 1/x diverges to infinity incredibly, unimaginably slowly. The first 100 terms sum to just over 5; the first 1000 to approx 7.5; the first million to approx 14.4; the first billion to approx. 21.3. Viewed in this way, the difference between a converging series and a series which only just diverges seems more understandable and more marginal, though the end result still lacks intuitive sense.

As a result of this final step, I finally came to terms with the nature of paradoxes within maths, and decided that all of the intuition that one tries to apply is much less rigorous than the rules of mathematics. More specifically, throughout my research I pinpointed multiple times where I felt that the brian's inability to conceptualise infinity was the main barrier between intuition and the proof: the automatic assumption that if something is endless, it must be significant until the end, leading one to believe that the horn does in fact have an infinite volume, but it is only when you accept that as values get so insignificant, they tend to a certain value, which in this case, was pi.