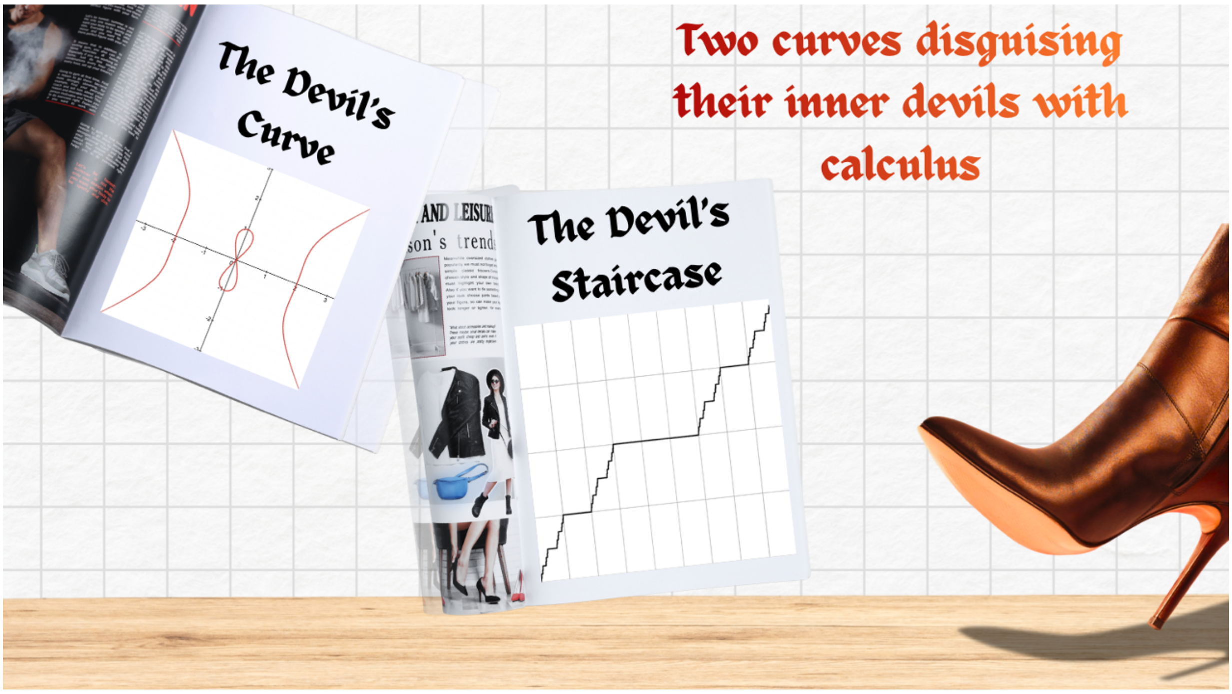
THE DEVILS WEAR CALCULUS

Exploring the Devil's Curve and the Devil's Staircase



1. Introduction

The coordinate plane's reputation is by far one of the most notorious in all mathematics. It certainly doesn't help its case then that it is home to two epitomes of evil: the Devil's Curve and the Devil's Staircase. These curves don't disguise their malice with makeup or designer clothes (as many movies about the fashion industry would have you to believe). Instead, they wear the malice behind a veil of numbers corresponding to one of the fundamental branches of mathematics: calculus.

This essay will explore these two curves and highlight an aspect which grants them a vital place in the spectrum of mathematics. These aspects are:

- The Devil's Curve being the smallest genus 2 curve
- The Devil's Staircase being continuous and increasing, despite having a derivative of 0 at nearly every point

Now let's grab our defenses (they are devils after all) and dive right in!

2. The Devil's Curve

2.1: Introducing the Curve

The Devil's Curve is referred to as *'la courbe du diable'* by French geometers and is recognized by the following Cartesian equation:

$$y^4 - a^2y^2 = x^4 - b^2x^2$$

This devil, like every other, disguises itself in variability. When the constants a and b are altered, the central hourglass part of the curve changes its orientation. For $\frac{a}{b} > 1$, the central hourglass is horizontal. This is shown in the figure below when $a = 3$ and $b = 2$:

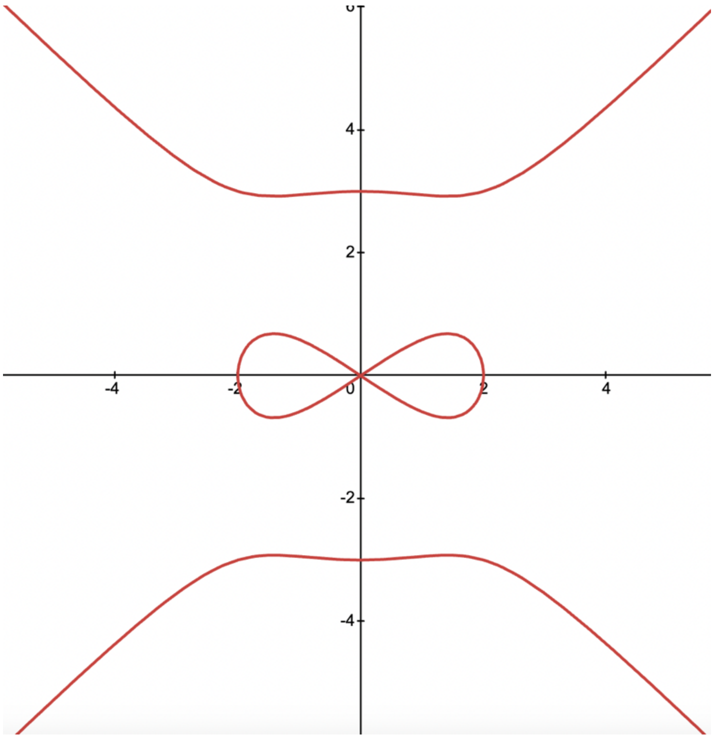


Figure 1: *The Devil's Curve when a = 3 and b = 2*

For $\frac{a}{b} < 1$, the central hourglass is vertical. This is seen when $a = 1$ and $b = 2$:

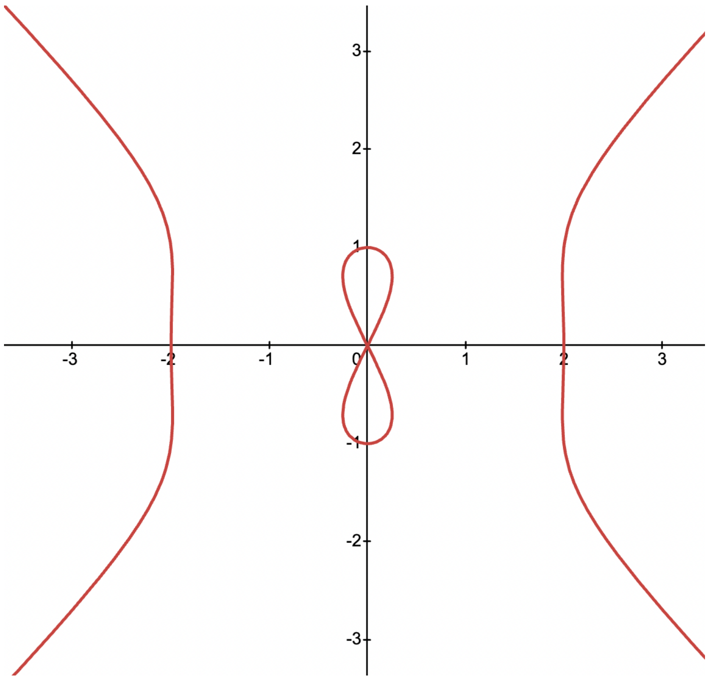


Figure 2: *The Devil's Curve when a = 1 and b = 2*

However, when a and b are equal, the hourglass suddenly collapses into a wheel:

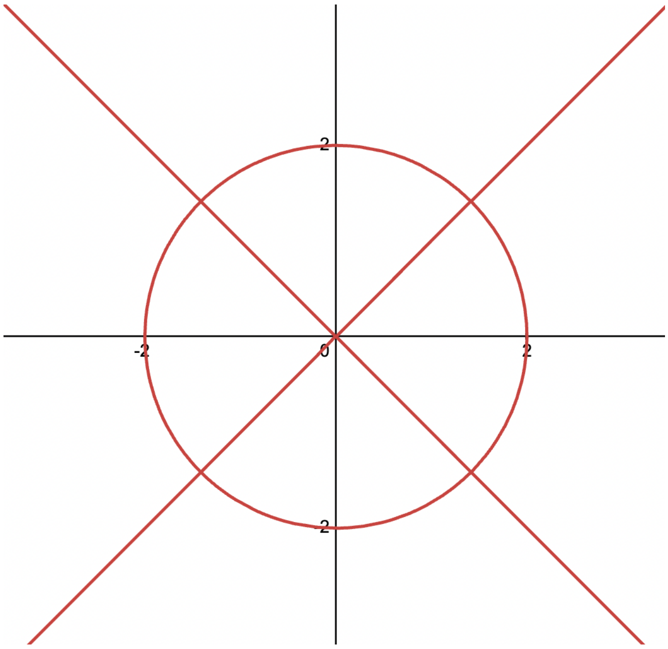


Figure 3: *The Devil's Curve when a = 2 and b = 2*

Interestingly, in all the curves in Figures 1, 2 and 3, the curve has a double point at the origin where it crosses itself twice.

The curve also lies on the polar plane. Its polar equation can be derived by substituting $x = r\cos\theta$ and $y = r\sin\theta$ into the equation of the curve:

$$(r\sin\theta)^4 - a^2(r\sin\theta)^2 = (r\cos\theta)^4 - b^2(r\cos\theta)^2$$

By rearranging, this becomes:

$$r^2\left(\sin^2\theta - \cos^2\theta\right) = a^2\sin^2\theta - b^2\cos^2\theta$$

2.2: Differentiating the Curve

I know all this rearrangement of equations looks overwhelming but trust me it's leading to somewhere. Here's a more interesting fact for the time being: the variable chaos of the curve is also transferred to its derivative. We can see this by using implicit differentiation on the curve's equation:

$$4y^3\frac{dy}{dx} - 4x^3 + 2ay\frac{dy}{dx} + 2bx = 0$$

$$\frac{dy}{dx} = \frac{4x^3 - 2bx}{4y^3 + \ 2ay}$$

$$\frac{dy}{dx} = \frac{2x\,(2x^2 - b)}{2y\,(2y^2 + a)}$$

Adding to its fickleness, the location of the curve's horizontal and vertical tangents changes according to the values of a and b. However, there positions can be generalized as follows:

For a horizontal tangent, the numerator of $\frac{dy}{dx}$ must equal 0:

$$2x\,(2x^2 - b) = 0$$

$$\Rightarrow x = 0 \ or \ x = \ \pm\sqrt{\frac{b}{2}}$$

For a vertical tangent, the denominator of $\frac{dy}{dx}$ must equal to 0, provided that the numerator is not 0:

$$2y\,(2y^2 + a) = 0$$

Now, let's move on to a more interesting aspect:

<u>2.3: Constructing the Curve with a Rectangular Hyperbola</u>

Yup that's right! The Devil's Curve and all its shape-shifting brilliance can be constructed from a single rectangular hyperbola. This hyperbola has the general equation $p^2 = \frac{a^2}{\cos 2\theta}$.

For this demonstration, I am using the Devil's Curve with $a = 2$ and $b = 4$:

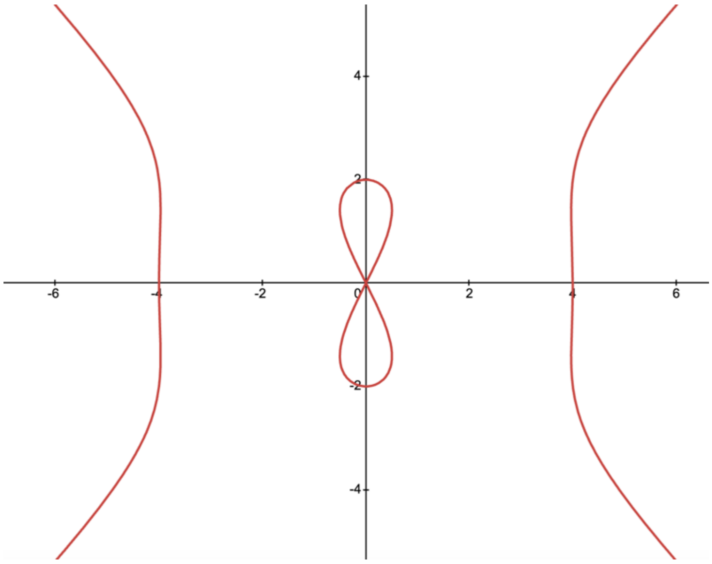


Figure 4: *The Devil's Curve when a=2 and b=4*

The equation of the hyperbola which matches the curvature of the curve is $y^2 = \sqrt{\frac{2.6^2}{\cos 2\theta}}$ (shown in blue in Figure 5 below). Let H be a point on this hyperbola and OH be the distance between the origin and point H. The line HN is perpendicular to the line OH. The infinite branches of the curve are loci of point M for which OM = ON. HN has a constant length for any points H or N considered.

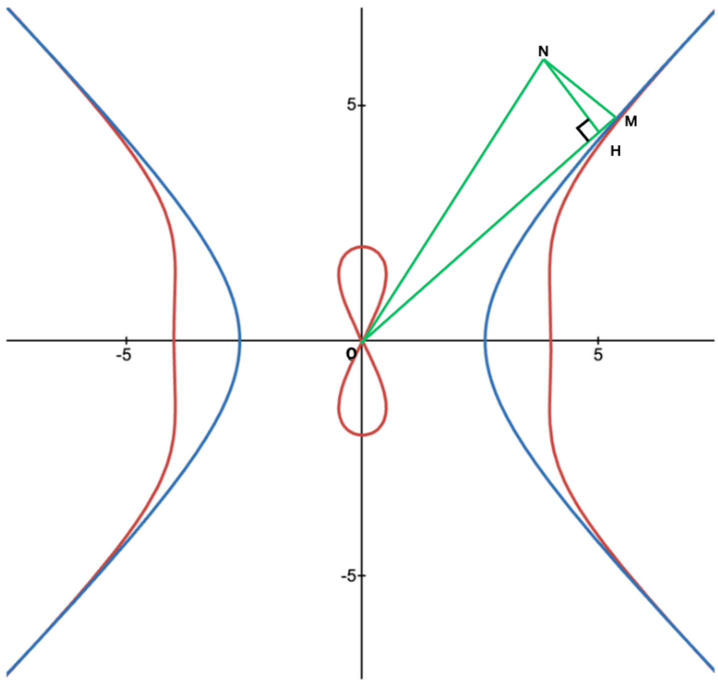


Figure 5: The curve in Figure 4 with the triangle ONM, point H, and the rectangular hyperbola

In fact, the central hourglass can also be constructed using the same method. As before, H is a point on this hyperbola. Point M, in this case, lies on the central hourglass. For any H or M considered, a right angle is formed at O:

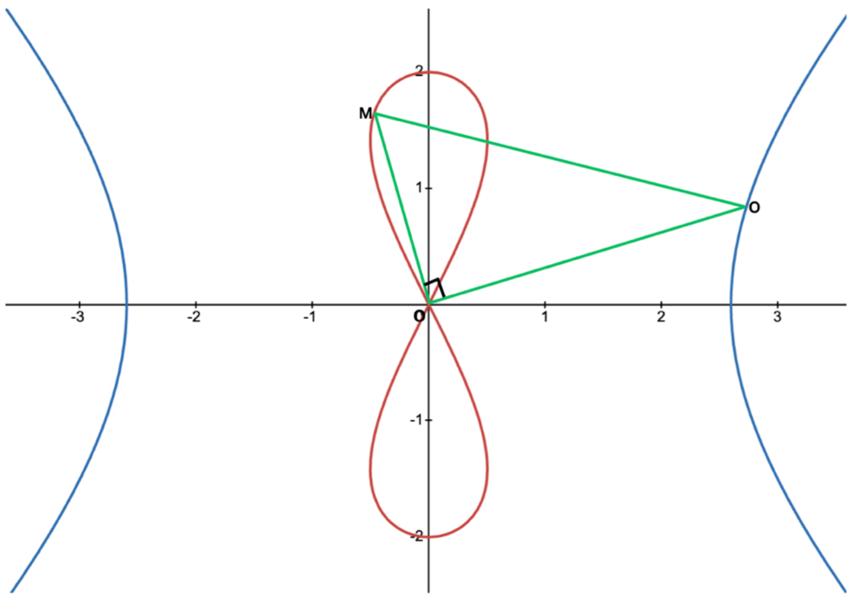


Figure 6: Constructing the central hourglass in Figure 4 using the method above

At this point in our discussion of the Devil's Curve, we've noticed:

- The curve has a central hourglass whose orientation changes according to the values of a and b
- The curve has a double point at the origin as it crosses itself twice
- The curve can be entirely constructed by a rectangular hyperbola.

All of this goes to prove, perhaps, the most interesting aspect of the Devil's Curve: it is the smallest curve with genus 2.

But I know what you must be wondering:

2.4: How does the Devil's Curve have genus 2?

Hopefully I read your mind and would be able to help you with the answer!

A genus 2 curve means a 'irreducible algebraic curve defined over an algebraically closed field k'. (Shaska) Topologically, a genus 2 curve has a double torus as its Reimann surface. A double torus looks like this:

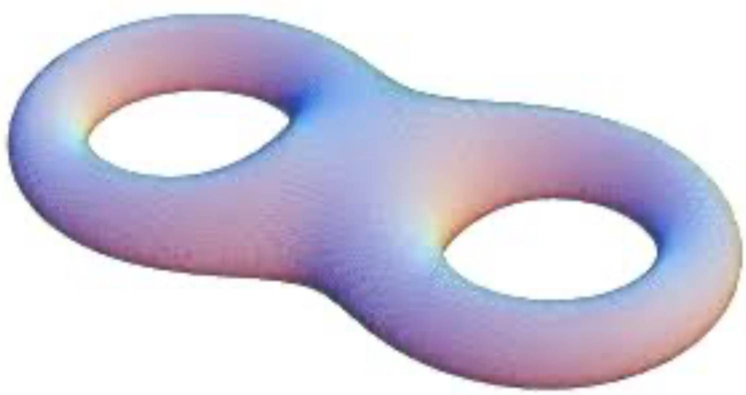


Figure 7: A Double Torus (Weisstein, Double Torus)

This does look an awful lot like the central hourglass of the Devil's Curve. Well, that's because, in projective space, the hourglass of the Devil's Curve actually has a double torus surface. This is, of course, when the singularity at the origin due to the double point is removed.

Therefore, the Devil's Curve is the smallest possible curve to have a genus 2. It definitely is a devil in disguise... but is it's not the only one! Don't forget, there's also the Devil's Staircase.

3. The Devil's Staircase

3.1: Introducing the Curve

The Devil's Staircase, at least in this context, does not lead to hell. In fact, it doesn't lead to anywhere! It's a rare mathematical function whose derivative is 0 at nearly all points, yet it still manages to increase from 0 to 1.

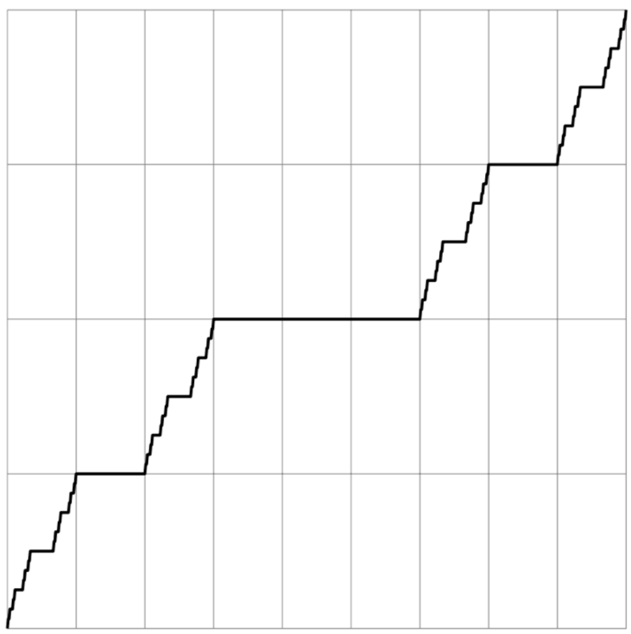


Figure 8: The Devil's Staircase (Girotti)

How it does this is not through witchcraft (this is mathematics after all!!!).

The Devil's Staircase is actually the cumulative distribution function (CDF) of the Cantor Function. It calculates the probability that X, a random variable, is **less than or equal** to a specific value x on the set. This can be expressed as $F(x) = P(X \leq x)$.

To understand how the Devil's Staircase is constructed, we need to have a closer look at the Cantor Function itself.

3.2: Linking the Devil's Staircase to the Cantor Function

This is the Cantor Function, which I will refer to as $F(x)$:

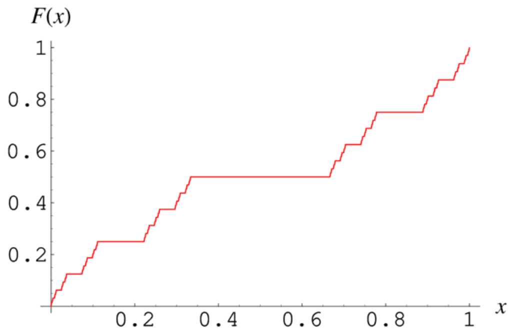


Figure 9: The Cantor Function

I know the resemblance is uncanny, but let's hold on before calling them both the same. I already mentioned that the Devil's Staircase is CDF of the Cantor Function! So, the Cantor Function is a special case of the Devil's Staircase.

The Cantor Function and the Devil's Staircase are constructed in the same way (surprise... surprise...). The construct the functions we need to express x in the ternary or base three number system. After that, we need to make the following changes to our ternary number:

- If the ternary number has the digit 1, replace the following digit with 0
- If the ternary number has the digit 2, replace that digit with 1

The flat lines on the set correspond to points where the digit 1 was removed.

This miraculously deletes the digit 2 from the set's existence, leaving a BINARY number which defines $F(x)$. How computational!

So now that we know how the Devil's Staircase is constructed, what's so special about it? Well, like I mentioned in Section 3.1, it increases monotonically without having a slope. It's like climbing a mountain without actually taking any steps (if that makes sense).

3.3: How does the Devil's Staircase increase without actually having a slope?

The answer is not a miracle. Instead, let's look at the Devil's Staircase again.

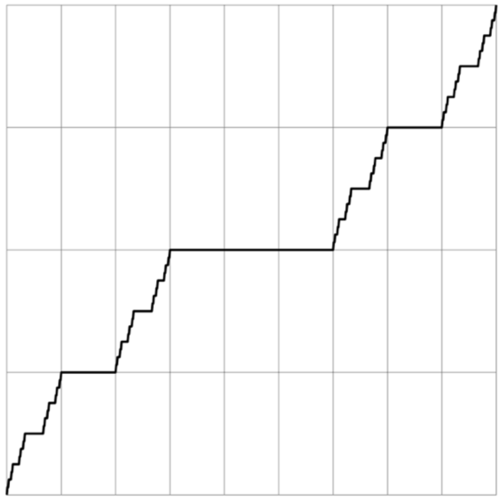


Figure 10: The Devil's Staircase (again...)

The flat sections obviously have a derivative of 0, but what about the points where the staircase seems to rise? Well, let me let you in on the biggest red herring in the history of mathematics: the staircase doesn't actually rise at all! The staircase appears to rise at intervals with points in the Cantor Function. Now, the Cantor Function has a measure of zero. This is because it is based on the Cantor Set which already has a measure of zero as the length of its pieces decreases infinitely by a factor of $\frac{2}{3}$ with each iteration. Therefore, the intervals where the staircase seems to rise do not actually have a measure.

So, this whole time, the Devil's Staircase never increases. It leads to nowhere (a bit of a cliffhanger in my opinion).

4. The Curves' Significance Outside the Coordinate Plane

Now that we have looked at our devilish curves, the calculus behind them, and what makes them special, there is only one last thing to explore. What if I told you that these curves are not just limited to the coordinate plane but can also be found in real life?

For the Devil's Curve, when $a = \sqrt{96}$ and $b = 10$, the curve forms an unusual shape:

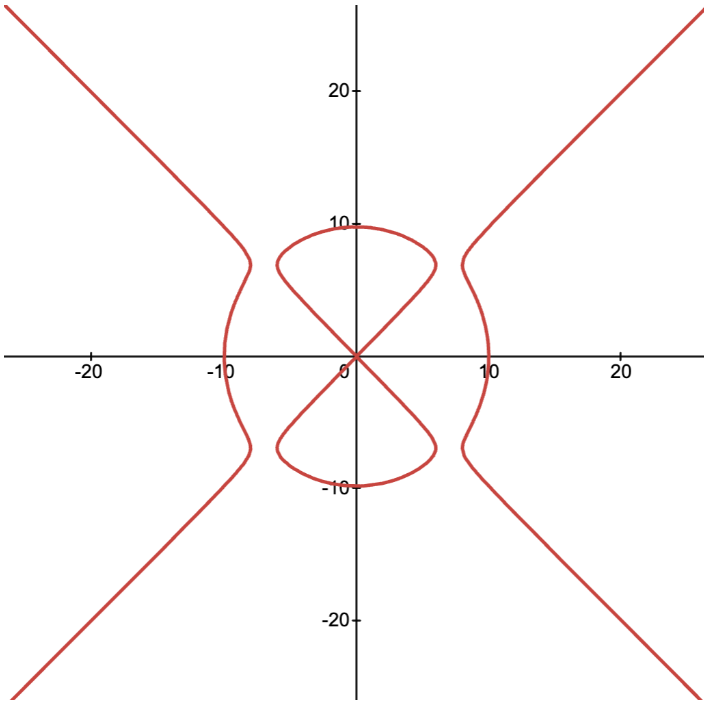


Figure 11: The Devil's Curve with $a = \sqrt{96}$ and $b = 10$

The hourglass resemblances a coil of wire rotating in between a magnetic field. For this reason, this version of the Devil's Curve is called the 'electric motor curve'.

The Devil's Staircase manifests itself in real life in a more invigorating manner. The 'West Highland Way' in Scotland actually takes the shape of an inverted Devil's Staircase, as can be seen in the map below:

Figure 12: A map of the 'West Highland Way' by Maximum Adventure

In fact, in the ‘West Highland Way’ lies a 548-meter-high point called ‘The Devil’s Staircase’ due to its steep incline.

Looking at these cases above, it is evident that the influence of Devil’s Curve and the Devil’s Staircase is not just restricted to the coordinate plane. Like a true devil, it doesn’t just mask itself in calculus, but also presents itself in the real world (maybe more often than we take notice).

5. A Final Note to the Reader

Now having analyzed the Devil’s Curve and the Devil’s Staircase, we have seen that these curves have achieved some impossible feats. The Devil’s Curve is the smallest curve with a genus 2. The Devil’s Staircase is the master of all paradoxes: it increases without increasing. They are brave devils indeed!

So, the next time you are bored by the coordinate plane, be careful that you haven’t summoned any of these devilish curves. Maybe things will start to seem more interesting!

6. Bibliography

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